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The Algebra of Formal Series III: Several Variables

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1. INTRODUCTION

It is the purpose of this paper to begin the development of the algebra of formal series in several variables. From the point of view of the present theory, the natural objects of study are not single series in n variables, but rather n-sets of series $(f_1, ..., f_n)$.

Let us take the variables to be $t_1, ..., t_n$. Then the counterpart of a delta series in one variable [that is, a series of the form $a_1 t + a_2 t^2 + \cdots$ with $a_1 \neq 0$] is a delta set $(f_1, ..., f_n)$, where f_j is of the form

$$f_j = a_{j,1} t_1 + \cdots + a_{j,n} t_n + g_j$$

with g_j being a power series whose terms are of degree at least two (or else $g_j = 0$) and where $(a_{j,i})$ is a nonsingular matrix of constants. These are precisely the sets of series which possess a compositional inverse. When $(a_{j,i})$ is the identity, we call the set $(f_1, ..., f_n)$ a diagonal delta set. To each delta set one can associate a sequence of series as in the single-variable case—a sequence which classically would be termed a sequence of "binomial type" in several variables.

For diagonal delta sets, we are able to generalize all the theory of the single-variable case. Thus we are able to study for the first time sequences of infinite series in several variables with both positive and negative exponents.

However, in the nondiagonal case, some serious difficulties arise in connection with negative exponents. It becomes somewhat of a problem even to define $(a_{j,1} t_1 + \cdots + a_{j,n} t_n)^{-1}$ in such a way that composition of series retains needed algebraic properties. Nevertheless, we have reason to believe that these difficulties are not insurmountable, and we feel close to a conclusion one way or the other about the existence and usefulness of sequences of series involving negative exponents. In the present paper

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we restrict our attention to nonnegative exponents for the nondiagonal case. In this setting the theory generalizes completely.

We have decided to postpone a detailed study of examples and applications of the present theory to a forthcoming paper. However, regretting somewhat the total absence of examples in our paper on the single-variable case, we have elected to give a few examples here, such as a multivariate version of the Abel and Laguerre polynomials.

One reason for the postponement of a discussion of examples is that in any single-variable case there may be many possible generalizations to several variables, and without specific motivation it is hard to know which way to proceed. Thus it seems pointless to pick arbitrary generalizations and compute examples for example sake.

In this paper we have merely scratched the surface of the vastly complicated theory of formal series in several variables. Not only do immediate questions remain concerning the present work, but many new directions present themselves. For example, we have not touched upon the combinatorial significance of any of the present results. Other directions include the study of the calculus of residues in several variables and a generalization to local rings. We hope in time to touch upon all of these.

2. The Algebras

Let K be a field of characteristic zero. Let Γ denote the vector space of all formal series in the variables $t_1, \ldots, t_n$ of the form

$$f = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} a_{i_1,\ldots,i_n} t^{i_1} \cdots t^{i_n},$$

where $a_{i_1,\ldots,i_n} \in K$, m is any integer, and where the asterisk indicates that the sum is a finite one. Under ordinary multiplication of formal series, Γ is an algebra.

The *degree* of $f \in \Gamma$ is the smallest integer m such that $a_{i_1,\ldots,i_n} \neq 0$ for some $i_1, \ldots, i_n$ with $i_1 + \cdots + i_n = m$. Notice that if $f, g \in \Gamma$, then $\deg fg = \deg f + \deg g$.

We let P be the algebra of all formal series in the variables $x_1, \ldots, x_n$ of the form

$$p = \sum_{v=-\infty}^{k} \sum_{j_1+\cdots+j_n=v}^{*} b_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n},$$

where $b_{j_1,\ldots,j_n} \in K$, k is any integer, and the inner sum is a finite one. The *degree* of p is the largest integer k such that $b_{j_1,\ldots,j_n} \neq 0$ for some $j_1, \ldots, j_n$ with $j_1 + \cdots + j_n = k$. For $p, q \in P$, we have $\deg pq = \deg p + \deg q$.

We put a topology on Γ by specifying that a sequence f_k in Γ converges to $f \in \Gamma$ if for any integer u_0 there exists an integer k_0 such that if $k \geqslant k_0$ then the coefficient of $t_1^{i_1} \cdots t_n^{i_n}$ in f_k equals the coefficient of $t_1^{i_1} \cdots t_n^{i_n}$ in f for all $i_1, \ldots, i_n$ with $i_1 + \cdots + i_n \leqslant u_0$. We put a similar topology on P. Namely, a sequence p_k in P converges to $p \in P$ if for any integer u_0 there exists an integer k_0 such that if $k \geqslant k_0$ then the coefficient of $t_1^{i_1} \cdots t_n^{i_n}$ in p_k equals the same coefficient in p for all $i_1 + \cdots + i_n \geqslant u_0$. Both Γ and P are topological algebras.

3. Diagonal Delta Sets

The set $(f_1, \ldots, f_n)$ is a *diagonal delta set* if

$$f_i = t_i + g_i$$

for $i = 1, \ldots, n$, where $g_i = 0$ or else g_i is a *power series* (that is, has no negative exponents) of degree at least two. Any element f_i of a delta set has a multiplicative inverse in Γ. For $g_i = 0$, this is clear. For $g_i \neq 0$ consider the series

$$\sum_{k \geqslant 0} (-1)^k \, t_i^{-1-k} g_i^{\,k}.$$

Since $\deg g_i \geqslant 2$, this series converges in Γ, and is therefore the multiplicative inverse of f_i.

If $f = \sum_{u=m}^{\infty} \sum^*_{i_1+\cdots+i_n=u} a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n}$ and if $g_1, \ldots, g_n \in \Gamma$ we define the *composition* of f with $g_1, \ldots, g_n$ as the series

$$f(g_1, \ldots, g_n) = \sum_{u=m}^{\infty} \sum^*_{i_1+\cdots+i_n=u} a_{i_1,\ldots,i_n} g_1^{i_1} \cdots g_n^{i_n},$$

provided the sum converges. If $(g_1, \ldots, g_n)$ is a diagonal delta set, then $\deg g_1^{i_1} \cdots g_n^{i_n} = i_1 + \cdots + i_n$ and so the sum will converge and $f(g_1, \ldots, g_n)$ is always defined. Moreover, if $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are diagonal delta sets, then $(f_1(g_1, \ldots, g_n), \ldots, f_n(g_1, \ldots, g_n))$ is a diagonal delta set. It is well known that any diagonal delta set has a *compositional inverse*, that is, a diagonal delta set $(\bar{f}_1, \ldots, \bar{f}_n)$ for which

$$f_i(\bar{f}_1, \ldots, \bar{f}_n) = t_i = \bar{f}_i(f_1, \ldots, f_n)$$

for all $i = 1, \ldots, n$.

We say that the set $p_{i_1,\ldots,i_n}$ in P, where $i_1, \ldots, i_n$ range over all integers, is a *strong sequence* if any $q \in P$ has a unique representation as a convergent sum

$$q = \sum_{u=-\infty}^{k} \sum_{i_1+\cdots+i_n=u}^{*} a_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n}.$$

4. An Action of Γ on P

We define an action of Γ on P. Let c_i be a sequence of nonzero elements of K for all integers i, and suppose $c_0 = 1$. We denote the action of $f \in \Gamma$ on $p \in P$ by

$$\langle f \mid p \rangle$$

and set

$$\langle t_1^{i_1} \cdots t_n^{i_n} \mid x_1^{j_1} \cdots x_n^{j_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1, j_1} \cdots \delta_{i_n, j_n},$$

where $\delta_{i,j}$ is the Kronecker delta. The action is extended to all $f \in \Gamma$ and $p \in P$. Thus if

$$f = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n}$$

and

$$p = \sum_{v=-\infty}^{k} \sum_{j_1+\cdots+j_n=v} b_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$$

we have

$$\langle f \mid p \rangle = \sum_{u=m}^{k} \sum_{i_1+\cdots+i_n=u}^{*} a_{i_1,\ldots,i_n} b_{i_1,\ldots,i_n} c_{i_1} \cdots c_{i_n}.$$

It is clear that $\langle f \mid p \rangle = 0$ if $\deg f > \deg p$.

Since $\langle f \mid x_1^{i_1} \cdots x_n^{i_n} \rangle = a_{i_1,\ldots,i_n} c_{i_1} \cdots c_{i_n}$ we have

$$f = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} \frac{\langle f \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{c_{i_1} \cdots c_{i_n}} t_1^{i_1} \cdots t_n^{i_n}.$$

Also,

$$p = \sum_{v=-\infty}^{k} \sum_{j_1+\cdots+j_n=v}^{*} \frac{\langle t_1^{j_1} \cdots t_n^{j_n} \mid p \rangle}{c_{j_1} \cdots c_{j_n}} x_1^{j_1} \cdots x_n^{j_n}.$$

From this it is clear that if $\langle f \mid p \rangle = 0$ for all $p \in P$, then $f = 0$ and if

$\langle f \mid p\rangle = 0$ for all $f \in \Gamma$, then $p = 0$. We call this the *spanning argument*. It is easy to verify

PROPOSITION 1. *If $f, g \in \Gamma$, then*

$$\langle fg \mid x_1^{i_1} \cdots x_n^{i_n}\rangle = \sum_{u=m}^{i_1+\cdots+i_n-k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{c_{i_1} \cdots c_{i_n}}{c_{j_1} \cdots c_{j_n} c_{i_1-j_1} \cdots c_{i_n-j_n}} \times \langle f \mid x_1^{j_1} \cdots x_n^{j_n}\rangle\langle g \mid x_1^{i_1-j_1} \cdots x_n^{i_n-j_n}\rangle,$$

where $m = \deg f$ and $k = \deg g$.

An induction argument gives

PROPOSITION 2. *If $f_1, \ldots, f_m \in \Gamma$, then*

$$\langle f_1 \cdots f_m \mid x_1^{i_1} \cdots x_n^{i_n}\rangle = \sum_{\substack{u_1 \geqslant \deg f_1 \\ \vdots \\ u_n \geqslant \deg f_m}} \sum_{\substack{j_1{}^1+\cdots+j_n{}^1=u_1 \\ + \quad + \\ \vdots \quad \vdots \\ + \quad + \\ j_1{}^m+\cdots+j_n{}^m=u_n \\ \| \quad \| \\ i_1 \quad i_n}} \frac{c_{i_1} \cdots c_{i_n}}{c_{j_1{}^1} \cdots c_{j_n{}^1} \cdots c_{j_1{}^m} \cdots c_{j_n{}^m}} \times \langle f_1 \mid x_1^{j_1{}^1} \cdots x_n^{j_n{}^1}\rangle \cdots \langle f_m \mid x_1^{j_1{}^m} \cdots x_n^{j_n{}^m}\rangle.$$

5. ASSOCIATED SEQUENCES

A strong sequence $p_{i_1,\ldots,i_n}$ is called the *associated sequence* for the diagonal delta set $(f_1, \ldots, f_n)$ if it satisfies

$$\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,\ldots,i_n}\rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$$

for all integers $j_1, \ldots, j_n$ and $i_1, \ldots, i_n$.

THEOREM 1. *Every diagonal delta set has a unique associated sequence.*

Proof. For the uniqueness, if $p_{i_1,\ldots,i_n}$ and $q_{i_1,\ldots,i_n}$ are both associated sequences then

$$t_1^{k_1} \cdots t_n^{k_n} = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} \frac{\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle}{c_{i_1} \cdots c_{i_n}} f_1^{i_1} \cdots f_n^{i_n}.$$

This follows from the fact that the right-hand sum converges, applying both sides to $p_{j_1,\ldots,j_n}$ gives equality, and that $p_{j_1,\ldots,j_n}$ is a strong sequence and therefore spans P. Thus

$$\begin{aligned}
&\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{j_1,\ldots,j_n}\rangle \\
&\quad = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} \frac{\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle}{c_{i_1} \cdots c_{i_n}} \langle f_1^{i_1} \cdots f_n^{i_n} \mid p_{j_1,\ldots,j_n}\rangle \\
&\quad = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} \frac{\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle}{c_{i_1} \cdots c_{i_n}} \langle f_1^{i_1} \cdots f_n^{i_n} \mid q_{j_1,\ldots,j_n}\rangle \\
&\quad = \langle t_1^{k_1} \cdots t_n^{k_n} \mid q_{j_1,\ldots,j_n}\rangle
\end{aligned}$$

and so $p_{j_1,\ldots,j_n} = q_{j_1,\ldots,j_n}$.

For the existence, the identity

$$\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle = \langle \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n}\rangle$$

defines a set $p_{i_1,\ldots,i_n}$ for which $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ and the only term in $p_{i_1,\ldots,i_n}$ of degree $i_1 + \cdots + i_n$ is a constant multiple of $t_1^{i_1} \cdots t_n^{i_n}$. Thus $p_{i_1,\ldots,i_n}$ is a strong sequence in P. Since

$$f_1^{j_1} \cdots f_n^{j_n} = \sum_{u=m}^{\infty} \sum_{k_1+\cdots+k_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{k_1} \cdots x_n^{k_n}\rangle}{c_{k_1} \cdots c_{k_n}} t_1^{k_1} \cdots t_n^{k_n},$$

we have

$$\begin{aligned}
&\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,\ldots,i_n}\rangle \\
&\quad = \sum_{u=m}^{\infty} \sum_{k_1+\cdots+k_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{k_1} \cdots x_n^{k_n}\rangle}{c_{k_1} \cdots c_{k_n}} \langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle \\
&\quad = \sum_{u=m}^{\infty} \sum_{k_1+\cdots+k_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{k_1} \cdots x_n^{k_n}\rangle}{c_{k_1} \cdots c_{k_n}} \langle \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n}\rangle \\
&\quad = \langle t_1^{j_1} \cdots t_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n}\rangle \\
&\quad = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}.
\end{aligned}$$

Note that if $p_{i_1,\ldots,i_n}$ is an associated sequence, then $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ and so

$$\sum_{v=-\infty}^{k} \sum_{i_1+\cdots+i_n=v}^{*} a_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n}$$

will always converge in P.

A convenient device for handling associated sequences is the transfer operator. If $p_{i_1,...,i_n}$ is an associated sequence the continuous linear operator λ on P defined by

$$\lambda x_1^{i_1} \cdots x_n^{i_n} = p_{i_1,...,i_n}$$

is called the *transfer operator* associated with $p_{i_1,...,i_n}$. Notice that λ is defined on all of P, and that λ is a bijection.

If μ is any linear operator on P, we define its *adjoint* μ^* as the unique linear operator on Γ defined by

$$\langle \mu^* f \mid x_1^{i_1} \cdots x_n^{i_n} \rangle = \langle f \mid \mu x_1^{i_1} \cdots x_n^{i_n} \rangle$$

for all $f \in \Gamma$.

THEOREM 2. *A linear operator λ on P is a transfer operator if and only if its adjoint λ^* is a continuous automorphism of Γ which maps delta sets to delta sets.*

Proof. It is clear that if λ is a transfer operator, then λ^* is linear, one-to-one and onto. The proof of Theorem 1 shows that if λ: $x_1^{i_1} \cdots x_n^{i_n} \to p_{i_1,...,i_n}$ is the associated sequence for $(f_1 ,..., f_n)$ then

$$\langle t_1^{k_1} \cdots t_n^{k_n} \mid p_{i_1,...,i_n} \rangle = \langle \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle.$$

Thus if $g \in \Gamma$ and

$$g = \sum_{u=m}^{\infty} \sum_{k_1+\cdots+k_n=u}^{*} a_{k_1,...,k_n} t_1^{k_1} \cdots t_n^{k_n}$$

we have

$$\begin{aligned}\langle \lambda^* g \mid x_1^{i_1} \cdots x_n^{i_n} \rangle &= \langle g \mid p_{i_1,...,i_n} \rangle \\ &= \sum_{u=m}^{\infty} \sum_{k_1+\cdots+k_n=u}^{*} a_{k_1,...,k_n} \langle \bar{f}_1^{k_1} \cdots \bar{f}_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle \\ &= \langle g(\bar{f}_1 ,..., \bar{f}_n) \mid x_1^{i_1} \cdots x_n^{i_n} \rangle.\end{aligned}$$

So $\lambda^* g = g(\bar{f}_1 ,..., \bar{f}_n)$ and λ^* is continuous, preserves products, and maps delta sets to delta sets.

For the converse, suppose μ^* is a continuous automorphism of Γ which maps delta sets to delta sets. Suppose $(f_1 ,..., f_n)$ is the delta set for which

$\mu^* f_1^{j_1} \cdots f_n^{j_n} = t_1^{j_1} \cdots t_n^{j_n}$. Then if $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1,\ldots,f_n)$ and λ: $x_1^{i_1} \cdots x_n^{i_n} \to p_{i_1,\ldots,i_n}$ is a transfer operator, we have

$$\lambda^* f_1^{j_1} \cdots f_n^{j_n} = f_1^{j_1}(\bar{f}_1,\ldots,\bar{f}_n) \cdots f_n^{j_n}(\bar{f}_1,\ldots,\bar{f}_n) = t_1^{j_1} \cdots t_n^{j_n}$$

and so $\lambda^* = \mu^*$.

The important properties of transfer operators are contained in

COROLLARY 1. (a) *If* λ: $x_1^{i_1} \cdots x_n^{i_n} \to p_{i_1,\ldots,i_n}$ *is a transfer operator, and* $p_{i_1,\ldots,i_n}$ *is associated to* $(f_1,\ldots,f_n)$, *then if* $g \in \Gamma$, *we have*

$$\lambda^* g = g(\bar{f}_1,\ldots,\bar{f}_n).$$

In particular,

$$\lambda^* f_1^{j_1} \cdots f_n^{j_n} = t_1^{j_1} \cdots t_n^{j_n}.$$

(b) *A transfer operator maps associated sequences to associated sequences.*

(c) *If* λ: $p_{i_1,\ldots,i_n} \to q_{i_1,\ldots,i_n}$ *is a linear operator, and* $p_{i_1,\ldots,i_n}$ *is associated to* $(f_1,\ldots,f_n)$, *and* $q_{i_1,\ldots,i_n}$ *is associated to* $(g_1,\ldots,g_n)$, *then* λ *is a transfer operator and*

$$\lambda^* g_1^{i_1} \cdots g_n^{i_n} = f_1^{i_1} \cdots f_n^{i_n}.$$

Proof. (a) Part (a) is proved in the proof of Theorem 2.

(b) Suppose λ: $x_1^{i_1} \cdots x_n^{i_n} \to p_{i_1,\ldots,i_n}$, and let $q_{i_1,\ldots,i_n}$ be the associated sequence for $(g_1,\ldots,g_n)$. Then $\langle (\lambda^{-1})^* g_1^{j_1} \cdots g_n^{j_n} \mid \lambda q_{i_1,\ldots,i_n} \rangle = \langle g_1^{j_1} \cdots g_n^{j_n} \mid q_{i_1,\ldots,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$ and so $\lambda q_{i_1,\ldots,i_n}$ is the associated sequence for $((\lambda^{-1})^* g_1,\ldots,(\lambda^{-1})^* g_n)$.

(c) We have $\langle \lambda^* g_1^{j_1} \cdots g_n^{j_n} \mid p_{i_1,\ldots,i_n} \rangle = \langle g_1^{j_1} \cdots g_n^{j_n} \mid q_{i_1,\ldots,i_n} \rangle = \langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,\ldots,i_n} \rangle$. Thus $\lambda^* g_1^{j_1} \cdots g_n^{j_n} = f_1^{j_1} \cdots f_n^{j_n}$. This implies that λ^* is a continuous automorphism of Γ mapping delta sets to delta sets, and so λ is a transfer operator by Theorem 2.

Suppose $p_{i_1,\ldots,i_n}$ and $q_{i_1,\ldots,i_n}$ are associated sequences and

$$p_{i_1,\ldots,i_n} = \sum_{u=-\infty}^{m} \sum_{j_1+\cdots+j_n=u}^{*} a_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}.$$

We define the *umbral composition* of $p_{i_1,\ldots,i_n}$ with $q_{i_1,\ldots,i_n}$ as the strong sequence

$$p_{i_1,\ldots,i_n}(\mathbf{q}) = \sum_{u=-\infty}^{m} \sum_{j_1+\cdots+j_n=u}^{*} a_{j_1,\ldots,j_n} q_{j_1,\ldots,j_n},$$

which converges in P.

The content of the next theorem is that the map which associates to each diagonal delta set its associated sequence is a group homomorphism from the group of diagonal delta sets under composition to the group of associated sequences under umbral composition.

THEOREM 3. *If* $(f_1, ..., f_n)$ *has associated sequence* $p_{i_1,...,i_n}$ *and* $(g_1, ..., g_n)$ *has associated sequence* $q_{i_1,...,i_n}$ *then* $(f_1(g_1, ..., g_n), ..., f_n(g_1, ..., g_n))$ *has associated sequence* $p_{i_1,...,i_n}(\mathbf{q})$.

Proof. If $\lambda: x_1^{i_1} \cdots x_n^{i_n} \to q_{i_1,...,i_n}$ is a transfer operator, then $\lambda p_{i_1,...,i_n} = p_{i_1,...,i_n}(\mathbf{q})$. Moreover, $(\lambda^{-1})^* f = f(g_1, ..., g_n)$ and so

$$\begin{aligned}&\langle f_1^{j_1}(g_1, ..., g_n) \cdots f_n^{j_n}(g_1, ..., g_n) \mid p_{i_1,...,i_n}(\mathbf{q})\rangle \\ &= \langle(\lambda^{-1})^* f_1^{j_1} \cdots f_n^{j_n} \mid \lambda p_{i_1,...,i_n}\rangle \\ &= \langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,...,i_n}\rangle \\ &= c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}\end{aligned}$$

and the theorem is proved.

If $(f_1, ..., f_n)$ is a diagonal delta set, and if $q_{i_1,...,i_n}$ is the associated sequence for the compositional inverse $(\bar{f}_1, ..., \bar{f}_n)$, then

$$q_{i_1,...,i_n} = \sum_{u=-\infty}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n}\rangle}{c_{j_1} \cdots c_{j_n}} x_1^{j_1} \cdots x_n^{j_n}.$$

We call $q_{i_1,...,i_n}$ the *conjugate sequence* for $(f_1, ..., f_n)$. Thus the conjugate sequence for a diagonal delta set is the associated sequence for its compositional inverse.

COROLLARY 2. *If* $p_{i_1,...,i_n}$ *is the associated sequence for* $(f_1, ..., f_n)$ *and* $q_{i_1,...,i_n}$ *is the conjugate sequence for* $(f_1, ..., f_n)$ *then*

$$p_{i_1,...,i_n}(\mathbf{q}) = x_1^{i_1} \cdots x_n^{i_n} = q_{i_1,...,i_n}(\mathbf{p}).$$

One of the key results of this section is

THEOREM 4 (Expansion Theorem). *Let* $(f_1, ..., f_n)$ *be a diagonal delta set with associated sequence* $p_{i_1,...,i_n}$. *Then if* $g \in \Gamma$, *we have*

$$g = \sum_{u=m}^{\infty} \sum_{i_1+\cdots+i_n=u}^{*} \frac{\langle g \mid p_{i_1,...,i_n}\rangle}{c_{i_1} \cdots c_{i_n}} f_1^{i_1} \cdots f_n^{i_n},$$

where $m = \deg g$.

Proof. It is clear that the sum on the right converges, and applying both sides to $p_{j_1,\ldots,j_n}$ gives equality. Therefore, the spanning argument proves the theorem.

The Expansion Theorem has some very important corollaries, which we examine next.

Suppose $u_1,\ldots, u_n$ are integers and $a_1,\ldots, a_n \in K$. The *evaluation series* $\epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n}$ in Γ is defined by

$$\epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} = \sum_{u=u_1+\cdots+u_n}^{\infty} \sum_{\substack{j_1+\cdots+j_n=u \\ j_i \geqslant u_i}} \frac{a_1^{j_1} \cdots a_n^{j_n}}{c_{j_1} \cdots c_{j_n}} t_1^{j_1} \cdots t_n^{j_n}.$$

This series has the property that

$$\begin{aligned} \langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle &= 0 \qquad &\text{if} \quad j_i < u_i \quad \text{for any } i, \\ &= a_1^{j_1} \cdots a_n^{j_n} \qquad &\text{if} \quad j_i \geqslant u_i \quad \text{for all } i. \end{aligned}$$

Moreover, if $\langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid p \rangle = 0$ for all evaluation series then $p = 0$.

Corollary 3. *If $p_{j_1,\ldots,j_n}$ is the associated sequence for $(f_1,\ldots,f_n)$ and if $q \in P$, then*

$$q = \sum_{u=-\infty}^{k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid q \rangle}{c_{j_1} \cdots c_{j_n}} p_{j_1,\ldots,j_n},$$

where $k = \deg q$.

Proof. From the Expansion Theorem we have

$$\epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} = \sum_{u=u_1+\cdots+u_n}^{\infty} \sum_{j_1+\cdots+j_n=u}^{*} \frac{\langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid p_{j_1,\ldots,j_n} \rangle}{c_{j_1} \cdots c_{j_n}} f_1^{j_1} \cdots f_n^{j_n};$$

applying q to both sides gives

$$\begin{aligned} \langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid q \rangle &= \sum_{u=u_1+\cdots+u_n}^{k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid q \rangle}{c_{j_1} \cdots c_{j_n}} \langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid p_{j_1,\ldots,j_n} \rangle \\ &= \left\langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \middle| \sum_{u=-\infty}^{k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid q \rangle}{c_{j_1} \cdots c_{j_n}} p_{j_1,\ldots,j_n} \right\rangle \end{aligned}$$

and since $u_1,\ldots, u_n$ and $a_1,\ldots, a_n$ are arbitrary the proof is complete.

We may use the Expansion Theorem to extend Proposition 1.

Corollary 4. *If $p_{i_1,\ldots,i_n}$ is an associated sequence and if $f, g \in \Gamma$, then*

$$\langle fg \mid p_{i_1,\ldots,i_n}\rangle = \sum_{u=m}^{i_1+\cdots+i_n-k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{c_{i_1}\cdots c_{i_n}}{c_{j_1}\cdots c_{j_n}c_{i_1-j_1}\cdots c_{i_n-j_n}} \times \langle f \mid p_{j_1,\ldots,j_n}\rangle\langle g \mid p_{i_1-j_1,\ldots,i_n-j_n}\rangle.$$

Corollary 4 has a converse.

PROPOSITION 3. *Suppose $p_{i_1,\ldots,i_n}$ is a strong sequence in P with the property that* $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ *and the only term in $p_{i_1,\ldots,i_n}$ of degree $i_1 + \cdots + i_n$ is a constant multiple of $x_1^{i_1}\cdots x_n^{i_n}$. If*

$$\langle fg \mid p_{i_1,\ldots,i_n}\rangle = \sum_{u=m}^{i_1+\cdots+i_n-k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{c_{i_1}\cdots c_{i_n}}{c_{j_1}\cdots c_{j_n}c_{i_1-j_1}\cdots c_{i_n-j_n}} \times \langle f \mid p_{j_1,\ldots,j_n}\rangle\langle g \mid p_{i_1-j_1,\ldots,i_n-j_n}\rangle$$

for all f and g in Γ with $m =$ $\deg f$ and $k = \deg g$ then $p_{i_1,\ldots,i_n}$ is an associated sequence.

Proof. For any $\alpha = 1,\ldots, n$ and any integer i_α we define the series f_{α,i_α} by

$$\langle f_{\alpha,i_\alpha} \mid p_{k_1,\ldots,k_n}\rangle = c_{i_\alpha}\delta_{i_\alpha,k_\alpha}\prod_{\beta\neq\alpha}\delta_{0,k_\beta}.$$

We would like first to show that $(f_{1,1},\ldots, f_{n,1})$ is a diagonal delta set. Now

$$\langle f_{\alpha,1} \mid p_{k_1,\ldots,k}\rangle = c_1\delta_{1,k_\alpha}\sum_{\beta\neq\alpha}\delta_{0,k_\beta}$$

and if $i_1 + \cdots + i_n \leqslant 0$, we can express $x_1^{i_1}\cdots x_n^{i_n}$ as an infinite sum of $p_{k_1,\ldots,k_n}$ which includes only those for which $k_1 + \cdots + k_n \leqslant 0$. Thus $\deg f_{\alpha,1} \geqslant 1$. Similarly, if $i_1 + \cdots + i_n = 1$ we may express $x_1^{i_1}\cdots x_n^{i_n}$ in terms of $p_{k_1,\ldots,k_n}$, where either $k_1 = i_1,\ldots, k_n = i_n$ or else $k_1 + \cdots + k_n < 1$ and so $\langle f_{\alpha,1} \mid x_1^{i_1}\cdots x_n^{i_n}\rangle = 0$ unless $i_\alpha = 1$ and $i_\beta = 0$ for all $\beta \neq \alpha$. Thus $\deg f_{\alpha,1} = 1$ and the only term in $f_{\alpha,1}$ of degree 1 is a constant multiple of t_α and so $(f_{1,1},\ldots, f_{n,1})$ is a diagonal delta set.

Now consider the product $f_{\alpha,l_\alpha}f_{\alpha,j_\alpha}$. We have

$$\begin{aligned}\langle f_{\alpha,l_\alpha}f_{\alpha,j_\alpha} \mid p_{k_1,\ldots,k_n}\rangle &= \sum_{u=l_\alpha}^{k_1+\cdots+k_n-j_\alpha} \sum_{i_1+\cdots+i_n=u}^{*} \frac{c_{k_1}\cdots c_{k_n}}{c_{i_1}\cdots c_{i_n}c_{k_1-i_1}\cdots c_{k_n-i_n}} \\ &\quad\times \langle f_{\alpha,l_\alpha} \mid p_{i_1,\ldots,i_n}\rangle\langle f_{\alpha,j_\alpha} \mid p_{k_1-i_1,\ldots,k_n-i_n}\rangle \\ &= c_{l_\alpha+j_\alpha}\delta_{l_\alpha+j_\alpha,k_\alpha}\prod_{\beta\neq\alpha}\delta_{0,k_\beta} \\ &= \langle f_{\alpha,l_\alpha+j_\alpha} \mid p_{k_1,\ldots,k_n}\rangle\end{aligned}$$

and the spanning argument implies that $f_{\alpha,l_\alpha} f_{\alpha,j_\alpha} = f_{\alpha,l_\alpha+j_\alpha}$ and so $f_{\alpha,l_\alpha} = f_{\alpha,1}^{l_\alpha}$. Finally,

$$\begin{aligned}\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{k_1,\ldots,k_n}\rangle &= \langle f_{1,j_1} \cdots f_{n,j_n} \mid p_{k_1,\ldots,k_n}\rangle \\ &= \sum_{u=j_1}^{k_1+\cdots+k_n-j_2-\cdots-j_n} \sum_{i_1+\cdots+i_n=u}^{*} \frac{c_{k_1} \cdots c_{k_n}}{c_{i_1} \cdots c_{i_n} c_{k_1-i_1} \cdots c_{k_n-i_n}} \\ &\quad \times \langle f_{1,j_1} \mid p_{i_1,\ldots,i_n}\rangle \langle f_{2,j_2} \cdots f_{n,j_n} \mid p_{i_1-i_1,\ldots,k_n-i_n}\rangle \\ &= \frac{c_{k_1}}{c_{k_1 j_1}} \langle f_{2,j_2} \cdots f_{n,j_n} \mid p_{k_1-j_1,k_2,\ldots,k_n}\rangle.\end{aligned}$$

Continuing in this way we obtain

$$\begin{aligned}\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{k_1,\ldots,k_n}\rangle &= \frac{c_{k_1} \cdots c_{k_{n-1}}}{c_{k_1-j_1} \cdots c_{k_{n-1}-j_{n-1}}} \langle f_{n,j_n} \mid p_{k_1-j_1,\ldots,k_{n-1}-j_{n-1},k_n}\rangle \\ &= c_{k_1} \cdots c_{k_n} \delta_{k_1,j_1} \cdots \delta_{k_n,j_n}\end{aligned}$$

and so $p_{k_1,\ldots,k_n}$ is the associated sequence for $(f_{1,1}, \ldots, f_{n,1})$.

In the very important special case that $c_k = k!$ for $k \geqslant 0$, it is a routine calculation to show that the evaluation series satisfies

$$\epsilon_{a_1,\ldots,a_n}^{0,\ldots,0} \epsilon_{b_1,\ldots,b_n}^{0,\ldots,0} = \epsilon_{a_1+b_1,\ldots,a_n+b_n}^{0,\ldots,0}.$$

For $p \in P$ we let $\tilde{p} \in P$ be defined by

$$\begin{aligned}\langle t_1^{i_1} \cdots t_n^{i_n} \mid \tilde{p}\rangle &= \langle t_1^{i_1} \cdots t_n^{i_n} \mid p\rangle \qquad \text{for} \quad i_1, \ldots, i_n \geqslant 0, \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{aligned}$$

Thus $\tilde{p}$ consists of that part of p which contains only nonnegative exponents. Then $\langle \epsilon_{a_1,\ldots,a_n}^{0,\ldots,0} \mid p\rangle = \langle \epsilon_{a_1,\ldots,a_n}^{0,\ldots,0} \mid \tilde{p}\rangle$ and we have by Corollary 4

$$\begin{aligned}\langle \epsilon_{a_1+b_1,\ldots,a_n+b_n}^{0,\ldots,0} \mid \tilde{p}_{i_1,\ldots,i_n}\rangle &= \sum_{u=0}^{i_1+\cdots+i_n} \sum_{\substack{j_1+\cdots+j_n=u \\ 0 \leqslant j_k \leqslant i_k}} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n} \\ &\quad \times \langle \epsilon_{a_1,\ldots,a_n}^{0,\ldots,0} \mid \tilde{p}_{j_1,\ldots,j_n}\rangle \langle \epsilon_{b_1,\ldots,b_n}^{0,\ldots,0} \mid \tilde{p}_{i_1-j_1,\ldots,i_n-j_n}\rangle.\end{aligned}$$

We may write this more suggestively as

$$\tilde{p}_{i_1,\ldots,i_n}(a_1 + b_1, \ldots, a_n + b_n)$$

$$= \sum_{u=0}^{i_n+\cdots+i_n} \sum_{\substack{j_1+\cdots+j_n=u \\ 0\leqslant j_n\leqslant i_k}} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n} \tilde{p}_{j_1,\ldots,j_n}(a_1, \ldots, a_n)\, \tilde{p}_{i_1-j_1,\ldots,i_n-j_n}(b_1, \ldots, b_n)$$

for all $a_i, b_i \in K$. We call this the *binomial identity*.

If $p_{i_1,\ldots,i_n}$ and $q_{i_1,\ldots,i_n}$ are associated sequences, and if

$$p_{i_1,\ldots,i_n} = \sum_{h=-\infty}^{k} \sum_{j_1+\cdots+j_n=u} a_{j_1,\ldots,j_n} q_{j_1,\ldots,j_n},$$

then the *connection-constants problem* is to determine the constants $a_{j_1,\ldots,j_n}$. One solution is given by

PROPOSITION 4. *If $p_{i_1,\ldots,i_n}$ is associated to $(f_1, \ldots, f_n)$ and $q_{i_1,\ldots,i_n}$ is associated to $(g_1, \ldots, g_n)$ and if*

$$p_{i_1,\ldots,i_n} = \sum_{u=-\infty} \sum_{j_1+\cdots+j_n=u} a_{j_1,\ldots,j_n} q_{j_1,\ldots,j_n} \qquad (*)$$

then the sequence

$$r_{i_1,\ldots,i_n} = \sum_{u=-\infty}^{k} \sum_{j_1+\cdots+j_n=u}^{*} a_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}$$

is the associated sequence for

$$(f_1(\bar{g}_1, \ldots, \bar{g}_n), \ldots, f_n(\bar{g}_1, \ldots, \bar{g}_n)).$$

6. ANOTHER ACTION OF Γ ON P

We wish to find a method of computing the associated sequence of a diagonal delta set. To this end we define another action of Γ on P, which we denote by juxtaposition. Our motivation in defining this action is the requirement that

$$\langle f \mid g x_1^{i_1} \cdots x_n^{i_n} \rangle = \langle fg \mid x_1^{i_1} \cdots x_n^{i_n} \rangle.$$

Expanding the right side using Proposition 1, the spanning argument forces us to take

$$gx_1^{i_1} \cdots x_n^{i_n} = \sum_{u=-\infty}^{i_1+\cdots+i_n-k} \sum_{j_1+\cdots+j_n=u}^{*} \frac{c_{i_1} \cdots c_{i_n}}{c_{j_1} \cdots c_{j_n} c_{i_1-j_1} \cdots c_{i_n-j_n}} \times \langle g \mid x_1^{i_1-j_1} \cdots x_n^{i_n-j_n} \rangle \, x_1^{j_1} \cdots x_n^{j_n}.$$

Thus

$$t_1^{k_1} \cdots t_n^{k_n} x_1^{i_1} \cdots x_n^{i_n} = \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-k_1} \cdots c_{i_n-k_n}} x_1^{i_1-k_1} \cdots x_n^{i_n-k_n}.$$

Moreover, we have

PROPOSITION 5. *If $f, g \in \Gamma$, then*

$$f(gx_1^{i_1} \cdots x_n^{i_n}) = (fg)\, x_1^{i_1} \cdots x_n^{i_n} = (gf)\, x_1^{i_1} \cdots x_n^{i_n} = g(fx_1^{i_1} \cdots x_n^{i_n}).$$

Proof. If $h \in \Gamma$, then

$$\begin{aligned} \langle h \mid f(gx_1^{i_1} \cdots x_n^{i_n}) \rangle &= \langle hf \mid gx_1^{i_1} \cdots x_n^{i_n} \rangle \\ &= \langle hfg \mid x_1^{i_1} \cdots x_n^{i_n} \rangle \\ &= \langle h \mid (fg)\, x_1^{i_1} \cdots x_n^{i_n} \rangle \end{aligned}$$

and so $f(gx_1^{i_1} \cdots x_n^{i_n}) = (fg)\, x_1^{i_1} \cdots x_n^{i_1}$. The rest is evident.

We can characterize associated sequences by means of this new action.

THEOREM 5. *A strong sequence $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1, \ldots, f_n)$ if and only if*

(1) $\langle t_1^0 \cdots t_n^0 \mid p_{i_1,\ldots,i_n} \rangle = \delta_{i_1,0} \cdots \delta_{i_n,0}$,

(2) $f_1^{j_1} \cdots f_n^{j_n} p_{i_1,\ldots,i_n} = \dfrac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} p_{i_1-j_1,\ldots,i_n-j_n}$.

Proof. Suppose $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1, \ldots, f_n)$. Then

$$\begin{aligned} \langle f_1^{k_1} \cdots f_n^{k_n} \mid f_1^{j_1} \cdots f_n^{j_n} p_{i_1,\ldots,i_n} \rangle &= \langle f_1^{k_1+j_1} \cdots f_n^{k_n+j_n} \mid p_{i_1,\ldots,i_n} \rangle \\ &= c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1+k_1} \cdots \delta_{i_n,j_n+k_n} \\ &= \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} \langle f_1^{k_1} \cdots f_n^{k_n} \mid p_{i_1-j_1,\ldots,i_n-j_n} \rangle \end{aligned}$$

and the spanning argument completes the proof. Conversely, if (1) and (2) hold, then

$$\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,\ldots,i_n} \rangle = \left\langle t_1{}^0 \cdots t_n{}^0 \,\middle|\, \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} \, p_{i_1-j_1,\ldots,i_n-j_n} \right\rangle$$
$$= c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}.$$

Theorem 5 and the Expansion Theorem imply

Corollary 5. *If $p_{i_1,\ldots,i_n}$ is an associated sequence, then for $f \in \Gamma$,*

$$\langle f \mid p_{i_1,\ldots,i_n} \rangle = \sum_{u=m}^{\infty} \sum_{j_1+\cdots+j_n=u}^{*} \frac{c_{i_1} \cdots c_{i_n}}{c_{j_1} \cdots c_{j_n} c_{i_1-j_1} \cdots c_{i_n-j_n}}$$
$$\times \langle f \mid p_{j_1,\ldots,j_n} \rangle \, p_{i_1-j_1,\ldots,i_n-j_n},$$

where $m = \deg f$.

In the important special case that

$$c_k = k! \qquad \text{for} \quad k \geqslant 0,$$
$$= \frac{(-1)^{k+1}}{(-k-1)!} \qquad \text{for} \quad k < 0$$

it is easy to show that if $i_1, \ldots, i_n < 0$, then

$$\epsilon_{b_1,\ldots,b_n}^{0,\ldots,0} x_1^{i_1} \cdots x_n^{i_n} = (x_1 + b_1)^{i_1} \cdots (x_n + b_n)^{i_n}.$$

Thus Corollary 4, with $f = \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n}$ and $g = \epsilon_{b_1,\ldots,b_n}^{0,\ldots,0}$ gives, for the associated sequence $p_{i_1,\ldots,i_n}$,

$$\langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid \epsilon_{b_1,\ldots,b_n}^{0,\ldots,0} p_{i_1,\ldots,i_n} \rangle = \sum_{u=u_1+\cdots+u_n}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u}^{j_k \leqslant i_k} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n}$$
$$\times \langle \epsilon_{a_1,\ldots,a_n}^{u_1,\ldots,u_n} \mid p_{j_1,\ldots,j_n} \rangle \langle \epsilon_{b_1,\ldots,b_n}^{0,\ldots,0} \mid p_{i_1-j_1,\ldots,i_n-j_n} \rangle$$

for all integers $u_1, \ldots, u_n$, all $a_k, b_k \in K$, and all negative integers $i_1, \ldots, i_n$. This may be written in the following suggestive form:

$$p_{i_1,\ldots,i_n}(x_1 + b_1, \ldots, x_n + b_n) = \sum_{u=-\infty}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u}^{j_k \leqslant i_k} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n}$$
$$\times p_{j_1,\ldots,j_n}(x_1, \ldots, x_n) \, \tilde{p}_{i_1-j_1,\ldots,i_n-j_n}(b_1, \ldots, b_n)$$

for all $b_1, \ldots, b_n \in K$ and all negative integers $i_1, \ldots, i_n$. We call this the *factor binomial identity*.

7. The Transfer Formula

In this section we derive a formula for the associated sequence to a diagonal delta set. For $j = 1,..., n$ let θ_j be the continuous operator on P defined by

$$\theta_j x_1^{i_1} \cdots x_n^{i_n} = \frac{(i_j + 1)\, c_{i_j}}{c_{i_j} + 1}\, x_1^{i_1} \cdots x_{j-1}^{i_{j-1}} x_j^{i_j+1} x_{j+1}^{i_{j+1}} \cdots x_n^{i_n}.$$

Then

$$\begin{aligned}\langle \theta_j^* t_1^{k_1} \cdots t_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle &= \frac{(i_j + 1)\, c_{i_j}}{c_{i_j} + 1} \langle t_1^{k_1} \cdots t_n^{k_n} \mid x_1^{i_1} \cdots x_j^{i_j+1} \cdots x_n^{i_n} \rangle \\ &= \langle k_j t_1^{k_1} \cdots t_j^{k_j-1} \cdots t_n^{k_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle \\ &= \left\langle \frac{\partial}{\partial t_j} t_1^{k_1} \cdots t_n^{k_n} \,\middle|\, x_1^{i_1} \cdots x_n^{i_n} \right\rangle,\end{aligned}$$

where $\partial/\partial t_j$ is the partial derivative operator on Γ. Thus $\theta_j^* = \partial/\partial t_j$.

If $f_1, ..., f_n \in \Gamma$, the *Jacobian* $\partial(f_1, ..., f_n)$ is the formal series

$$\partial(f_1, ..., f_n) = \det\left(\frac{\partial}{\partial t_j} f_i\right).$$

Theorem 6 (Transfer Formula). *If $p_{i_1,...,i_n}$ is the associated sequence for the diagonal delta set $(f_1, ..., f_n)$ then*

$$p_{i_1,...,i_n} = \frac{c_{i_1} \cdots c_{i_n}}{c_{-1}^n}\, \partial(f_1, ..., f_n)\, f_1^{-1-i_1} \cdots f_n^{-1-i_n} x_1^{-1} \cdots x_n^{-1}.$$

Proof. We will show that the right-hand side satisfies the conditions of Theorem 5. It is easy to see that the right-hand side is a strong sequence, and condition 2 is straightforward.

We must show that

$$\langle \partial(f_1, ..., f_n)\, f_1^{-1-i_n} \cdots f_n^{-1-i_n} \mid x_1^{-1} \cdots x_n^{-1} \rangle = c_{-1}^n \delta_{i_1,0} \cdots \delta_{i_n,0}.$$

Since $f_j = t_j - g_j$ we have

$$f_j^{-1-i_j} = \sum_{k_j \geq 0} \binom{i_j + k_j}{k_j} g_j^{k_j} t_j^{-1-i_j-k_j}$$

and writing ∂ for $\partial(f_1, ..., f_n)$ and D_j for $\partial/\partial t_j$,

$$\langle \partial f_1^{-1-i_1} \cdots f_n^{-1-i_n} \mid x_1^{-1} \cdots x_n^{-1} \rangle$$

$$= \left\langle \partial \sum_{k_1,\ldots,k_n \geq 0} \binom{i_1+k_1}{k_1} \cdots \binom{i_n+k_n}{k_n} g_1^{k_1} \cdots g_n^{k_n} \,\middle|\, t_1^{-1-i_1-k_1} \cdots t_n^{-1-i_n-k_n} x_1^{-1} \cdots x_n^{-1} \right\rangle$$

$$= \left\langle \partial \sum_{k_1,\ldots,k_n \geq 0} \binom{i_1+k_1}{k_1} \cdots \binom{i_n+k_n}{k_n} g_1^{k_1} \cdots g_n^{k_n} \,\middle|\, \frac{c_{-1}^n}{c_{i_1+k_1} \cdots c_{i_n+k_n}} x_1^{i_1+k_1} \cdots x_n^{i_n+k_n} \right\rangle$$

$$= \left\langle \partial \sum_{k_1,\ldots,k_n \geq 0} \frac{g_1^{k_1}}{k_1!} \cdots \frac{g_n^{k_n}}{k_n!} \,\middle|\, \frac{c_{-1}^n}{c_{i_1} \cdots c_{i_n}} \theta_1^{k_1} \cdots \theta_n^{k_n} x_1^{i_1} \cdots x_n^{i_n} \right\rangle$$

$$= \frac{c_{-1}^n}{c_{i_1} \cdots c_{i_n}} \left\langle \sum_{k_1,\ldots,k_n \geq 0} D_{k_1} \cdots D_{k_n} \left(\partial \frac{g_1^{k_1}}{k_1!} \cdots \frac{g_n^{k_n}}{k_n!} \right) \,\middle|\, x_1^{i_1} \cdots x_n^{i_n} \right\rangle.$$

So if we write $g_i^{k_i}/k_i! = h_i^{k_i}$, we are left with showing that

$$\sum_{k_1,\ldots,k_n \geq 0} D_{k_1} \cdots D_{k_n} [\partial h_1^{k_1} \cdots h_n^{k_n}] = 1.$$

This fact has been proved by S. A. Joni but we repeat it here for the sake of completeness. We have $\partial = \det(\delta_{i,j} - D_i g_j)$ and so

$$\partial h_1^{k_1} \cdots h_n^{k_n} = \det(\delta_{i,j} h_j^{k_j} - D_i h_j^{k_j+1}).$$

If $A = \{1,\ldots, n\}$, then this determinant is equal to (see Muir, p. 109)

$$\sum_{a \subseteq A} (-1)^{|a|} \left(\prod_{i \in A-a} h_i^{k_i} \right) \det(D_i h_j^{k_j+1})_{(i,j) \in a \times a}$$

and so we must show that

$$\sum_{a \subseteq A} (-1)^{|a|} \sum_{k_1,\ldots,k_n \geq 0} D_1^{k_1} \cdots D_n^{k_n} \left[\left(\prod_{i \in A-a} h_i^{k_i} \right) \det(D_i h_j^{k_j+1})_{(i,j) \in a \times a} \right] = 1.$$

Note that if $k_i = 0$, then $h_i^{k_i} = 1$. If we think of $b \subseteq A$ as the set for which $k_i = 0$ if $i \in A - b$, we may write the above sum as

$$\sum_{a \subseteq A} (-1)^{|a|} \sum_{\substack{k_\alpha \geq 0 \\ \alpha \in a}} \sum_{b \supseteq a} \sum_{\substack{k_\beta \geq 1 \\ \beta \in b-a}} D_1^{k_1} \cdots D_n^{k_n} \left[\left(\prod_{i \in A-a} h_i^{k_i} \right) \det(D_i h_j^{k_j+1})_{(i,j) \in a \times a} \right],$$

where if $a = b = \varnothing$ the sum is 1. This equals

$$\sum_{b\subseteq A} \sum_{a\subseteq b} (-1)^{|a|} \sum_{\substack{k_\alpha \geq 0 \\ \alpha\in b}} \Big(\prod_{i\in a} D_i^{k_i}\Big)\Big(\prod_{i\in b-a} D_i^{k_i+1}\Big)\Big[\Big(\prod_{i\in b-a} h_i^{k_i+1}\Big) \det(D_i h_j^{k_j+1})_{(i,j)\in a\times a}\Big]$$

$$= \sum_{b\subseteq A} \sum_{\substack{k_\alpha \geq 0 \\ \alpha\in b}} \Big(\prod_{i\in b} D_i^{k_i}\Big) \sum_{a\subseteq b} (-1)^{|a|} \Big(\prod_{i\in b-a} D_i\Big)\Big[\Big(\prod_{i\in b-a} h_i^{k_i+1}\Big) \det(D_i h_j^{k_j+1})_{(i,j)\in a\times a}\Big].$$

The following lemma will then complete the proof.

LEMMA. *If $l_i \in \Gamma$ for $i = 1,..., n$, then for b, a nonempty subset of A,*

$$\sum_{a\subseteq b} (-1)^{|a|} \Big(\prod_{i\in b-a} D_i\Big)\Big[\Big(\prod_{i\in b-a} l_i\Big) \det(D_i l_j)_{(i,j)\in a\times a}\Big] = 0.$$

Proof of lemma. We may assume that $b = \{1,..., m\}$. After all differentiation is performed, each term is of the form

$$\pm(B_1 l_1) \cdots (B_m l_m),$$

where $\{B_1 ,..., B_m\}$ forms a partition (with possibly $B_i = \varnothing$) of $\{D_1 ,..., D_m\}$. In fact, for a fixed set $a \subseteq b$ and for each permutation σ of a, the determinant produces terms of this form for which $D_{\sigma(i)} \in B_i$ for all $i \in a$; that is, terms of the form

$$(-1)^{|a|} (-1)^\sigma \Big(\prod_{i\in a} c_i D_{\sigma(i)} l_i\Big)\Big(\prod_{j\in b-a} c_j l_j\Big),$$

where $\{c_1 ,..., c_m\}$ is a partition of $\{D_j\}_{j\in b-a}$. Moreover, all contributions from set a are of this form for some σ. However, as the set a varies, we count each term of the above form more than once. Consider a fixed partition $\{B_1 ,..., B_m\}$ of $\{D_1 ,..., D_m\}$ and the corresponding expression

$$(B_1 l_1) \cdots (B_m l_m).$$

If there are r cycles $\alpha_1 ,..., \alpha_r$ in b for which $D_{\alpha_i(j)} \in B_j$ for all j for which $\alpha_i(j)$ is defined, then this expression is counted once for each product of any of the cycles $\alpha_1 ,..., \alpha_r$. The corresponding set a is the union of the cycles involved in the product. Moreover, if σ is the product of k cycles, then $(-1)^{|a|}(-1)^\sigma = (-1)^k$. Thus the above expression is counted

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^k = 0$$

times, and the lemma is proved.

8. Sheffer Sequences

A large number of sequences occurring in the literature are not associated sequences, but are closely related to them. If $p_{i_1,\ldots,i_n}$ is an associated sequence in P and if g is a series of degree 0 in Γ, then the sequence

$$s_{i_1,\ldots,i_n} = g p_{i_1,\ldots,i_n}$$

is called the *Sheffer sequence for* $p_{i_1,\ldots,i_n}$ *relative to* g.

The following lemma will help characterize Sheffer sequences.

Lemma. *Suppose L is a linear operator on P with the property that*

$$fLp = Lfp$$

for all $f \in \Gamma$ and $p \in P$. Then there exists a series $l \in \Gamma$ for which

$$lp = Lp$$

for all $p \in P$.

Proof. We have

$$\begin{aligned} \langle L^*(f) \mid p \rangle &= \langle f \mid Lp \rangle \\ &= \langle 1 \mid fLp \rangle \\ &= \langle 1 \mid Lfp \rangle \\ &= \langle L^*(1) \mid fp \rangle \\ &= \langle L^*(1) f \mid p \rangle \end{aligned}$$

and so

$$L^*(1) f = L^*(f).$$

Then if we set $l = L^*(1)$, we have

$$\langle f \mid lp \rangle = \langle fl \mid p \rangle = \langle L^*(f) \mid p \rangle = \langle f \mid Lp \rangle$$

and so

$$lp = Lp.$$

We can now characterize Sheffer sequences.

Theorem 7. *A strong sequence $s_{i_1,\ldots,i_n}$ in P is a Sheffer sequence if and only if there exists a diagonal delta set $(f_1,\ldots,f_n)$ for which*

$$f_1^{j_1} \cdots f_n^{j_n} s_{i_1,\ldots,i_n} = \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} s_{i_1-j_1,\ldots,i_n-j_n} .$$

Proof. If $s_{i_1,\ldots,i_n}$ is a Sheffer sequence then there exists an associated sequence $p_{i_1,\ldots,i_n}$ for which

$$s_{i_1,\ldots,i_n} = g p_{i_1,\ldots,i_n} .$$

Then if $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1, \ldots, f_n)$ we have

$$\begin{aligned} f_1^{j_1} \cdots f_n^{j_n} s_{i_1,\ldots,i_n} &= f_1^{j_1} \cdots f_n^{j_n} g p_{i_1,\ldots,i_n} \\ &= g f_1^{j_1} \cdots f_n^{j_n} p_{i_1,\ldots,i_n} \\ &= g \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} p_{i_1-j_1,\ldots,i_n-j_n} \\ &= \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} s_{i_1-j_1,\ldots,i_n-j_n} . \end{aligned}$$

For the converse, define the continuous linear operator L by

$$L p_{i_1,\ldots,i_n} = s_{i_1,\ldots,i_n} ,$$

where $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1, \ldots, f_n)$. Then

$$\begin{aligned} f_1^{j_1} \cdots f_n^{j_n} L p_{i_1,\ldots,i_n} &= f_1^{j_1} \cdots f_n^{j_n} s_{i_1,\ldots,i_n} \\ &= \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1-j_1} \cdots c_{i_n-j_n}} s_{i_1-j_1,\ldots,i_n-j_n} \\ &= L f_1^{j_1} \cdots f_n^{j_n} p_{i_1,\ldots,i_n} \end{aligned}$$

and so

$$Lf = fL$$

for all $f \in \Gamma$. The lemma then implies that there exists $l \in \Gamma$ for which

$$l p_{i_1,\ldots,i_n} = s_{i_1,\ldots,i_n} .$$

Since $\deg l = 0$ the sequence $s_{i_1,\ldots,i_n}$ is a Sheffer sequence.

9. Delta Sets

Suppose $f_1, \ldots, f_n \in \Gamma$ are of the form

$$f_j = a_{j,1} t_1 + \cdots + a_{j,n} t_n + g_j ,$$

where $a_{j,i} \in K$, $g_j = 0$ or else g_j is a power series of degree at least two, and $\det a_{j,i} \neq 0$. Then it is well known that $(f_1, ..., f_n)$ has a compositional inverse $(\bar{f}_1, ..., \bar{f}_n)$, which is of the same form as $(f_1, ..., f_n)$. We call such sets $(f_1, ..., f_n)$ *delta sets.*

Unfortunately, our work up to now is not general enough to deal with delta sets. This is mainly because an element f_j of a delta set does not necessarily have a multiplicative inverse in Γ. It is possible to generalize the algebra Γ and thereby introduce a multiplicative inverse. However, all attempts made so far to do this seem to produce more difficulties than they eliminate. We are forced therefore to restrict our considerations rather than to extend them.

Let $\Lambda \subseteq \Gamma$ be the algebra of all formal power series in the variables $t_1, ..., t_n$. Thus if $f \in \Lambda$ we have

$$f = \sum_{u=m}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} a_{i_1,\ldots,i_n} t_1^{i_n} \cdots t_n^{i_n},$$

where m is a nonnegative integer, and the inner sum is automatically a finite one. Let $R \subseteq P$ be the algebra of all polynomials in the variables $x_1, ..., x_n$. Thus $p \in R$ may be written

$$p = \sum_{v=0}^{k} \sum_{\substack{j_1+\cdots+j_n=v \\ j_i \geqslant 0}} b_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}.$$

Most of the definitions and results of the previous sections carry over to the subalgebras Λ and R. Therefore, we will proceed informally, giving proofs only when there is a significant deviation from the earlier theory.

We keep the same definitions of *degree, strong sequence* in R, *composition* in Λ and *umbral composition* in R. Moreover, we keep the same definition of the action of Γ on P as described in Section 4. In other words, if $f \in \Lambda$ and $p \in R$, we think of the action $\langle f \mid p \rangle$ as the one defined for $f \in \Gamma$ and $p \in P$. Thus

$$f = \sum_{u=m}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} \frac{\langle f \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{c_{i_1} \cdots c_{i_n}} t_1^{i_1} \cdots t_n^{i_n}$$

and

$$p = \sum_{v=0}^{k} \sum_{\substack{j_1+\cdots+j_n=v \\ j_i \geqslant 0}} \frac{\langle t_1^{j_1} \cdots t_n^{j_n} \mid p \rangle}{c_{j_1} \cdots c_{j_n}} x_1^{j_1} \cdots x_n^{j_n}.$$

The *spanning arguments* still hold for Λ and R, and so does Proposition 1.

However, the action of Γ on P described in Section 6 needs some modification. We take

$$t_1^{k_1} \cdots t_n^{k_n} x_1^{i_1} \cdots x_n^{i_n} = \frac{c_{i_1} \cdots c_{i_n}}{c_{k_1-i_1} \cdots c_{k_n-i_n}} x_1^{i_1-k_1} \cdots x_n^{i_n-k_n} \quad \text{if} \quad i_j \geqslant k_j \text{ for all } j,$$
$$= 0 \qquad \text{otherwise,}$$

and extend this to all of Λ and R. If $g \in \Lambda$ we have

$$g x_1^{j_1} \cdots x_n^{j_n} = \sum_{u=m}^{j_1+\cdots+j_n} \sum_{\substack{i_1+\cdots+i_n=u \\ i_k \geqslant 0}}^{i_k \leqslant j_k} \frac{c_{j_1} \cdots c_{j_n}}{c_{i_1} \cdots c_{i_n} c_{j_1-i_1} \cdots c_{j_n-i_n}} \times \langle g \mid x_1^{i_1-j_1} \cdots x_n^{i_n-j_n} \rangle x_1^{j_1} \cdots x_n^{j_n}$$

as well as

$$\langle f \mid g x_1^{i_1} \cdots x_n^{i_n} \rangle = \langle fg \mid x_1^{i_1} \cdots x_n^{i_n} \rangle$$

and

$$f g x_1^{i_1} \cdots x_n^{i_n} = g f x_1^{i_n} \cdots x_n^{i_n}.$$

10. Associated Sequences for Delta Sets

The *associated sequence* for a delta set $(f_1, ..., f_n)$ in Λ is the strong sequence $p_{i_1,...,i_n}$ satisfying

$$\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,...,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$$

for all nonnegative integers $j_1, ..., j_n$ and $i_1, ..., i_n$. Our first task is to show that the associated sequence exists and is unique.

Suppose $(f_1, ..., f_n)$ is a delta set. Then since $\det a_{i,j} \neq 0$ we conclude that any $g \in \Lambda$ can be written as a sum,

$$g = \sum_{u=m}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} a_{i_1,...,i_n} f_1^{i_1} \cdots f_n^{i_n}, \tag{*}$$

for some $m \geqslant 0$ and $a_{i_1,...,i_n} \in K$.

Now suppose $p_{i_1,...,i_n}$ is a *set* of elements of Λ, where $i_1, ..., i_n$ range over all nonnegative integers. Thus $p_{i_1,...,i_n}$ need not be a strong sequence. Let $p_{i_1,...,i_n}$ have the property that

$$\langle f_1^{j_1} \cdots f_n^{j_n} \mid p_{i_1,...,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n} .$$

Then if we apply both sides of (*) to $p_{j_1,\ldots,j_n}$ we obtain

$$a_{j_1,\ldots,j_n} = \frac{\langle g \mid p_{j_1,\ldots,j_n}\rangle}{c_{j_1}\cdots c_{j_n}}$$

and so

$$g = \sum_{u=m}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u\\ i_j\geqslant 0}} \frac{\langle g \mid p_{i_1,\ldots,i_n}\rangle}{c_{i_1}\cdots c_{i_n}} f_1^{i_1}\cdots f_n^{i_n}.$$

We would like to conclude that the set $p_{i_1,\ldots,i_n}$ is a strong sequence in R. That is, that each $q \in R$ can be written as a unique sum

$$q = \sum_{u=0}^{k} \sum_{\substack{i_1+\cdots+i_j=u\\ i_j\geqslant 0}} a_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n}$$

for some $k \geqslant 0$ and $a_{i_1,\ldots,i_n} \in K$. Recall that the evaluation series $\epsilon^{0,\ldots,0}_{a_1,\ldots,a_n}$ is defined by

$$\langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid x_1^{i_1}\cdots x_n^{i_n}\rangle = a_1^{i_1}\cdots a_n^{i_n}$$

for all $i_1,\ldots,i_n \geqslant 0$. It is clear that if $p, q \in R$ and $\langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid p\rangle = \langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid q\rangle$ for all $a_1,\ldots,a_n \in K$ then $p = q$. Since

$$\epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} = \sum_{u=0}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u\\ i_j\geqslant 0}} \frac{\langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid p_{i_1,\ldots,i_n}\rangle}{c_{i_1}\cdots c_{i_n}} f_1^{i_1}\cdots f_n^{i_n}$$

we conclude that for any $q \in R$ with $\deg q = k$,

$$\langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid q\rangle = \sum_{u=0}^{k} \sum_{\substack{i_1+\cdots+i_n=u\\ i_j\geqslant 0}} \frac{\langle f_1^{i_1}\cdots f_n^{i_n} \mid q\rangle}{c_{i_1}\cdots c_{i_n}} \langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \mid p_{i_1,\ldots,i_n}\rangle$$

$$= \left\langle \epsilon^{0,\ldots,0}_{a_1,\ldots,a_n} \,\middle|\, \sum_{u=0} \sum_{\substack{i_1+\cdots+i_n=u\\ i_j\geqslant 0}} \frac{\langle f_1^{i_1}\cdots f_n^{i_n} \mid q\rangle}{c_{i_1}\cdots c_{i_n}} p_{i_1,\ldots,i_n}\right\rangle$$

and so

$$q = \sum_{u=0}^{k} \sum_{\substack{i_1+\cdots+i_n=u\\ i_j\geqslant 0}} \frac{\langle f_1^{i_1}\cdots f_n^{i_n} \mid q\rangle}{c_{i_1}\cdots c_{i_n}} p_{i_1,\ldots,i_n}.$$

Moreover, if

$$q = \sum_{u=0} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} a_{i_1,\ldots,i_n} p_{i_1,\ldots,i_n}$$

then by applying $f_1^{j_1} \cdots f_n^{j_n}$ to both sides we see that

$$a_{j_1,\ldots,j_n} = \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid q \rangle}{c_{j_1} \cdots c_{j_n}}$$

and thus the coefficients are uniquely determined. So $p_{i_1,\ldots,i_n}$ is a strong sequence.

THEOREM 8. *Every delta set has a unique associated sequence.*

Proof. The uniqueness proof is the same as that in Theorem 1. The identity

$$\langle t_1^{j_1} \cdots t_n^{j_n} \mid p_{i_1,\ldots,i_n} \rangle = \langle \bar{f}_1^{j_1} \cdots \bar{f}_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle$$

defines a set $p_{i_1,\ldots,i_n}$ in R and as in the proof of Theorem 1 we have

$$\langle f^{j_1} \cdots f^{j_n} \mid p_{i_1,\ldots,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}.$$

By previous remarks the set $p_{i_1,\ldots,i_n}$ is a strong sequence in R and therefore is the associated sequence for $(f_1, \ldots, f_n)$.

It is clear from the proof of Theorem 8 that $\deg p_{i_1,\ldots,i_n} \leqslant i_1 + \cdots + i_n$. To see that $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ we must show that $p_{i_1,\ldots,i_n}$ has a term of the form $t_1^{j_1} \cdots t_n^{j_n}$ for which $j_1 + \cdots + j_n = i_1 + \cdots + i_n$. That is, we must show that $\langle t_1^{j_1} \cdots t_n^{j_n} \mid p_{i_1,\ldots,i_n} \rangle = \langle \bar{f}_1^{j_1} \cdots \bar{f}_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle$ is different from zero for some $j_1 + \cdots + j_n = i_1 + \cdots + i_n$.

Clearly, we may assume that

$$f_i = a_{i,1} t_1 + \cdots + a_{i,n} t_n.$$

Then we have

$$t_i = a_{i,1} \bar{f}_1 + \cdots + a_{i,n} \bar{f}_n$$

and so

$$\begin{aligned} t_1^{i_1} \cdots t_n^{i_n} &= \prod_{i=1}^{n} (a_{i,1} \bar{f}_1 + \cdots + a_{i,n} \bar{f}_n)^{i_n} \\ &= \sum_{u_1+\cdots+u_n=i_1+\cdots+i_n} \alpha_{u_1,\ldots,u_n} \bar{f}_1^{u_1} \cdots \bar{f}_n^{u_n} \end{aligned}$$

for some constants $\alpha_{u_1,\ldots,u_n}$. Thus for some $u_1,\ldots,u_n$ with $u_1 + \cdots + u_n = i_1 + \cdots + i_n$ it must be true that $\bar{f}_1^{u_1} \cdots \bar{f}_n^{u_n}$ contains a term of the form $t_1^{i_1} \cdots t_n^{i_n}$.

The *transfer operator* associated with $p_{i_1,\ldots,i_n}$ is the linear operator λ defined by

$$\lambda x_1^{i_1} \cdots x_n^{i_n} = p_{i_1,\ldots,i_n}$$

and the analog of Theorem 2 and its corollaries hold for Λ and R.

THEOREM 9. *A linear operator λ on R is a transfer operator if and only if its adjoint λ^* is a continuous automorphism of Λ which maps delta sets to delta sets.*

COROLLARY 6. (a) *If $\lambda x_1^{i_1} \cdots x_n^{i_n} \to p_{i_1,\ldots,i_n}$ is a transfer operator and $p_{i_1,\ldots,i_n}$ is associated to $(f_1,\ldots,f_n)$, then if $g \in \Lambda$,*

$$\lambda^* g = g(\bar{f}_1,\ldots,\bar{f}_n).$$

In particular,

$$\lambda^* f_1^{j_1} \cdots f_n^{j_n} = t_1^{j_1} \cdots t_n^{j_n}.$$

(b) *A transfer operator maps associated sequences to associated sequences.*

(c) *If $\lambda: p_{i_1,\ldots,i_n} \to q_{i_1,\ldots,i_n}$ is a linear operator, and $p_{i_1,\ldots,i_n}$ is associated to $(f_1,\ldots,f_n)$ and $q_{i_1,\ldots,i_n}$ is associated to $(g_1,\ldots,g_n)$ then λ is a transfer operator and*

$$\lambda^* g_1^{i_1} \cdots g_n^{i_n} = f_1^{i_1} \cdots f_n^{i_n}.$$

THEOREM 10. *If $(f_1,\ldots,f_n)$ has associated sequence $p_{i_1,\ldots,i_n}$ and $(g_1,\ldots,g_n)$ has associated sequence $q_{i_1,\ldots,i_n}$, then $(f_1(g_1,\ldots,g_n),\ldots,f_n(g_1,\ldots,g_n))$ has associated sequence $p_{i_1,\ldots,i_n}(\mathbf{q})$.*

The *conjugate sequence* for the delta set $(f_1,\ldots,f_n)$ is the associated sequence for $(\bar{f}_1,\ldots,\bar{f}_n)$ and so equals

$$q_{i_1,\ldots,i_n} = \sum_{u=0}^{i_1+\cdots+i_n} \sum_{\substack{j_1+\cdots+j_n=u \\ j_i \geqslant 0}} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{c_{j_1} \cdots c_{j_n}} x_1^{j_1} \cdots x_n^{j_n}.$$

COROLLARY 7. *If $p_{i_1,\ldots,i_n}$ and $q_{i_1,\ldots,i_n}$ are the associated and conjugate sequences for $(f_1,\ldots,f_n)$, then*

$$p_{i_1,\ldots,i_n}(\mathbf{q}) = x_1^{i_1} \cdots x_n^{i_n} = q_{i_1,\ldots,i_n}(\mathbf{p}).$$

We also have the all important Expansion Theorem and its corollaries.

THEOREM 11 (Expansion Theorem). *Let* $(f_1,\dots,f_n)$ *be a delta set with associated sequence* $p_{i_1,\dots,i_n}$. *Then if* $g \in \Gamma$, *we have*

$$g = \sum_{u=m}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} \frac{\langle g \mid p_{i_1,\dots,i_n}\rangle}{c_{i_1}\cdots c_{i_n}} f_1^{i_1}\cdots f_n^{i_n},$$

where $m = \deg g$.

COROLLARY 8. *If* $p_{i_1,\dots,i_n}$ *is the associated sequence for* $(f_1,\dots,f_n)$ *and if* $q \in P$, *then*

$$q = \sum_{u=0}^{k} \sum_{\substack{j_1+\cdots+j_n=u \\ j_i \geqslant 0}} \frac{\langle f_1^{j_1}\cdots f_n^{j_n} \mid q\rangle}{c_{j_1}\cdots c_{j_n}} p_{j_1,\dots,j_n},$$

where $k = \deg q$.

The Expansion Theorem gives us the generating function of the associated sequence.

COROLLARY 9. *If* $p_{i_1,\dots,i_n}$ *is the associated sequence for* $(f_1,\dots,f_n)$, *then*

$$\epsilon^{0,\dots,0}_{y_1,\dots,y_n}(\bar{f}_1,\dots,\bar{f}_n) = \sum_{u=0}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} p_{i_1,\dots,i_n}(y_1,\dots,y_n)\frac{t_1^{i_1}}{c_{i_1}}\cdots\frac{t_n^{i_n}}{c_{i_n}}.$$

For $c_n = n!$, we also obtain a formula for the compositional inverse of a delta set.

COROLLARY 10. *If* $p_{i_1,\dots,i_n}$ *is the associated sequence for a delta set* $(f_1,\dots,f_n)$, *with compositional inverse* $(\bar{f}_1,\dots,\bar{f}_n)$, *then if* $c_n = n!$,

$$\bar{f}_j = \sum_{u=0}^{\infty} \sum_{\substack{i_1+\cdots+i_n=u \\ i_j \geqslant 0}} \frac{\partial}{\partial x_j} p_{i_1,\dots,i_n}(0,\dots,0)\frac{t_1^{i_1}}{i_1!}\cdots\frac{t_n^{i_n}}{i_n!}.$$

COROLLARY 11. *If* $p_{i_1,\dots,i_n}$ *is an associated sequence and if* $f, g \in \Lambda$, *then*

$$\langle fg \mid p_{i_1,\dots,i_n}\rangle = \sum_{u=m}^{i_1+\cdots+i_n-k} \sum_{\substack{j_1+\cdots+j_n=u \\ j_i \geqslant 0}} \frac{c_{i_1}\cdots c_{i_n}}{c_{j_1}\cdots c_{j_n}c_{i_1-j_1}\cdots c_{i_n-j_n}} \times \langle f \mid p_{j_1,\dots,j_n}\rangle\langle g \mid p_{i_1-j_1,\dots,i_n-j_n}\rangle,$$

where $m = \deg f$ *and* $k = \deg g$.

The next proposition is proved in a manner similar to the proof of Proposition 3.

PROPOSITION 6. *Suppose* $p_{i_1,\ldots,i_n}$ *is a strong sequence in* R, *with* $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$. *If*

$$\langle fg \mid p_{i,\ldots,i} \rangle = \sum_{u=m}^{i_1+\cdots+i_n-k} \sum_{\substack{j_1+\cdots+j_n=u \\ j_k \geqslant 0}} \frac{c_{i_1} \cdots c_{i_n}}{c_{j_1} \cdots c_{j_n} c_{i_1-j_1} \cdots c_{i_n-j_n}} \times \langle f \mid p_{j_1,\ldots,j_n} \rangle \langle g \mid p_{i_1-j_1,\ldots,i_n-j_n} \rangle$$

for all $f, g \in \Lambda$ *with* $m = \deg f$ *and* $k = \deg g$, *then* $p_{i_1,\ldots,i_n}$ *is an associated sequence.*

In the special case that $c_k = k!$ for $k \geqslant 0$, Corollary 9 allows us to derive the binomial identity in R, namely, if $p_{i_1,\ldots,i_n}$ is an associated sequence in R, we have

$$p_{i_1,\ldots,i_n}(a_1 + b_1, \ldots, a_n + b_n)$$

$$\sum_{u=0}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u} \binom{i_1}{j_1} \cdots \binom{i_n}{j_n} p_{j_1,\ldots,j_n}(a_1, \ldots, a_n)\, p_{i_1-j_1,\ldots,i_n-j_n}(b_1, \ldots, b_n)$$

for all $a_i, b_i \in K$.

For the algebras Λ and R, the binomial identity is enough to guarantee that a strong sequence $p_{i_1,\ldots,i_n}$ with $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ is an associated sequence.

PROPOSITION 7. *If* $p_{i_1,\ldots,i_n}$ *is a strong sequence in* R *with* $\deg p_{i_1,\ldots,i_n} = i_1 + \cdots + i_n$ *satisfying the binomial identity, then it is an associated sequence.*

Proof. We need only verify the hypothesis of Proposition 6. Let $R[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the vector space of polynomials in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. If $f \in \Lambda$, then f induces a linear operator f_x on $R[x_1, \ldots, x_n, y_1, \ldots, y_n]$ as follows. If $p = \sum a_{i_1,\ldots,i_n,j_1,\ldots,j_n} x_1^{i_1} \cdots x_n^{i_n} y_1^{i_1} \cdots y_n^{i_n}$, then

$$f_x p = \sum a_{i_1,\ldots,i_n,j_1,\ldots,j_n} \langle f \mid x_1^{i_1} \cdots x_n^{i_n} \rangle\, y_1^{j_1} \cdots y_n^{j_n}.$$

Similarly, the operator f_y is defined by

$$f_y p = \sum a_{i_1,\ldots,i_n,j_1,\ldots,j_n} \langle f \mid x_1^{j_1} \cdots x_n^{j_n} \rangle\, x_1^{i_1} \cdots x_n^{i_n}.$$

In this notation Proposition 1 becomes

$$\langle fg \mid x_1^{k_1} \cdots x_n^{k_n} \rangle = f_x g_y \sum_{u \geqslant 0} \sum_{\substack{j_1 + \cdots + j_n = u \\ 0 \leqslant j_l \leqslant k_l}} \binom{k_1}{j_1} \cdots \binom{k_n}{j_n} x_1^{j_1} \cdots x_n^{j_n} y_1^{k_1 - j_1} \cdots y_n^{k_n - j_n}$$

$$= f_x g_y (x_1 + y_1)^{k_1} \cdots (x_n + y_n)^{k_n}.$$

Thus for any $p \in R$, we may write

$$\langle fg \mid p \rangle = f_x g_y p(x_1 + y_1, \ldots, x_n + y_n).$$

Choosing $p = p_{i_1, \ldots, i_n}$ and using the binomial identity gives the result.

The connection–constants problem has a similar solution in R,

PROPOSITION 8. *If $p_{i_1, \ldots, i_n}$ is associated to $(f_1, \ldots, f_n)$ and $q_{i_1, \ldots, i_n}$ is associated to $(g_1, \ldots, g_n)$ and if*

$$p_{i_1, \ldots, i_n} = \sum_{u=0}^{k} \sum_{\substack{j_1 + \cdots + j_n = u \\ j_i \geqslant 0}} a_{j_1, \ldots, j_n} q_{j_1, \ldots, j_n}$$

then the sequence

$$r_{i_1, \ldots, i_n} = \sum_{u=0}^{k} \sum_{\substack{j_1 + \cdots + j_n = u \\ j_i \geqslant 0}} a_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}$$

is the associated sequence for

$$(f_1(\bar{g}_1, \ldots, \bar{g}_n), \ldots, f_n(\bar{g}_1, \ldots, \bar{g}_n)).$$

The associated sequence of a delta set can be characterized as before.

THEOREM 12. *A strong sequence $p_{i_1, \ldots, i_n}$ is the associated sequence for $(f_1, \ldots, f_n)$ if and only if*

(1) $\langle t_1^0 \cdots t_n^0 \mid p_{i_1, \ldots, i_n} \rangle = \delta_{i_1, 0} \cdots \delta_{i_n, 0}$,

(2) $f_1^{j_1} \cdots f_n^{j_n} p_{i_1, \ldots, i_n} = \dfrac{c_{i_1} \cdots c_{i_n}}{c_{i_1 - j_1} \cdots c_{i_n - j_n}} p_{i_1 - j_1, \ldots, i_n - j_n}$ *for* $j_k \leqslant i_k$,

$k = 1, \ldots, n$.

COROLLARY 12. *If $p_{i_1,\ldots,i_n}$ is an associated sequence, then for $f \in \Gamma$ we have*

$$\langle f \mid p_{i_1,\ldots,i_n}\rangle = \sum_{u=m}^{\infty} \sum_{\substack{j_1+\cdots+j_n=u\\ j_i\geqslant 0}} \frac{c_{i_1}\cdots c_{i_n}}{c_{j_1}\cdots c_{j_n} c_{i_1-j_1}\cdots c_{i_n-j_n}} \times \langle f \mid p_{j_1,\ldots,j_n}\rangle\, p_{i_1-j_1,\ldots,i_n-j_n}.$$

Finally, we remark that the notion and properties of the Sheffer sequence in R are analogous to those in P.

11. THE TRANSFER FORMULA FOR DELTA SETS

The most elementary delta sets are of the form

$$f_i = a_{i,1}t_1 + \cdots + a_{i,n}t_n,$$

where $\det(a_{i,j}) \neq 0$. If $(b_{i,j})$ is the inverse matrix to $(a_{i,j})$, then

$$\bar{f}_i = b_{i,1}t_1 + \cdots + b_{i,n}t_n.$$

The associated sequence $p_{i_1,\ldots,i_n}$ for $(f_1,\ldots,f_n)$ is the conjugate sequence for $(\bar{f}_1,\ldots,\bar{f}_n)$ and so

$$p_{i_1,\ldots,i_n} = \sum_{u=0}^{i_1+\cdots+i_n} \sum_{\substack{j_1+\cdots+j_n=u\\ j_i\geqslant 0}} \frac{\langle \bar{f}_1^{j_1}\cdots \bar{f}_n^{j_n} \mid x_1^{i_1}\cdots x_n^{i_n}\rangle}{c_{j_1}\cdots c_{j_n}} x_1^{j_1}\cdots x_n^{j_n}$$

$$= \sum_{\substack{j_1+\cdots+j_n\leqslant i_1+\cdots+i_n\\ j_i\geqslant 0}} \frac{\langle \bar{f}_1^{j_1}\cdots \bar{f}_n^{j_n} \mid x_1^{i_1}\cdots x_n^{i_n}\rangle}{c_{j_1}\cdots c_{j_n}} x_1^{j_1}\cdots x_n^{j_n}.$$

In the special case where $c_k = k!$ for all $k \geqslant 0$, we can simplify this considerably. We have

$$\bar{f}_i^{j_i} = (b_{i,1}t_1 + \cdots + b_{i,n}t_n)^{j_i}$$

$$= \sum_{u_1^i+\cdots+u_n^i=j_i} \binom{j_i}{u_1^i,\ldots,u_n^i} (b_{i,1}t_1)^{u_1^i}\cdots(b_{i,n}t_n)^{u_n^i}$$

so

$$\bar{f}_1^{j_1}\cdots\bar{f}_n^{j_n} = \sum_{\substack{u_1^1+\cdots+u_n^1=j_1\\ \vdots\\ u_1^n+\cdots+u_n^n=j_n}} \binom{j_1}{u_1^1,\ldots,u_n^1}\cdots\binom{j_n}{u_1^n,\ldots,u_n^n} \times (b_{1,1}^{u_1^1}\cdots b_{1,n}^{u_n^1})\cdots(b_{n,1}^{u_1^n}\cdots b_{n,n}^{u_n^n})\, t_1^{u_1^1+\cdots+u_1^n}\cdots t_n^{u_n^1+\cdots+u_n^n}.$$

Applying this to $x_1^{i_1} \cdots x_n^{i_n}$ we must have

$$\begin{aligned} i_1 &= u_1{}^1 + \cdots + u_1{}^n \\ &\ \ \vdots \\ i_n &= u_n{}^1 + \cdots + u_n{}^n \end{aligned}$$

as well as

$$j_1 + \cdots + j_n = i_1 + \cdots + i_n .$$

Thus we obtain

$$\begin{aligned} &\frac{\langle \bar{f}_1^{j_1} \cdots \bar{f}_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{j_1! \cdots j_n!} \\ &= \frac{i_1! \cdots i_n!}{j_1! \cdots j_n!} \sum_{\substack{u_1{}^1 + \cdots + u_n{}^1 = j_1 \\ + \qquad\quad + \\ \vdots \qquad\quad \vdots \\ + \qquad\quad + \\ u_1{}^n + \cdots + u_n{}^n = j_n \\ \parallel \qquad\quad \parallel \\ i_1 \qquad\quad i_n}} \binom{j_1}{u_1{}^1, \dots, u_n{}^1} \cdots \binom{j_n}{u_1{}^n, \dots, u_n{}^n} \\ &\qquad \times b_{1,1}^{u_1{}^1} \cdots b_{1,n}^{u_n{}^1} \cdots b_{n,1}^{u_1{}^n} \cdots b_{n,n}^{u_n{}^n} \end{aligned}$$

and

$$\begin{aligned} p_{i_1,\dots,i_n} &= \sum_{\substack{j_1 + \cdots + j_n = i_1 + \cdots + i_n \\ j_i \geqslant 0}} \frac{\langle \bar{f}_1^{j_1} \cdots \bar{f}_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{j_1! \cdots j_n!} x_1^{j_1} \cdots x_n^{j_n} \\ &= \sum_{\substack{j_1 + \cdots + j_n = i_1 + \cdots + i_n \\ j_i \geqslant 0}} \sum_{\substack{u_1{}^1 + \cdots + u_n{}^1 = j_1 \\ + \qquad\quad + \\ \vdots \qquad\quad \vdots \\ + \qquad\quad + \\ u_1{}^n + \cdots + u_n{}^n = j_n \\ \parallel \qquad\quad \parallel \\ i_1 \qquad\quad i_n}} \frac{i_1! \cdots i_n!}{u_1{}^1! \cdots u_n{}^1! \cdots u_1{}^n! \cdots u_n{}^n!} \\ &\qquad \times b_{1,1}^{u_1{}^1} \cdots b_{1,n}^{u_n{}^1} \cdots b_{n,n}^{u_n{}^n} x_1^{u_1{}^1 + \cdots + u_n{}^1} \cdots x_n^{u_1{}^n + \cdots + u_n{}^n} \\ &= \sum_{\substack{u_1{}^1 + \cdots + u_1{}^n = i_1 \\ \vdots \\ u_n{}^1 + \cdots + u_n{}^n = i_n}} \binom{i_2}{u\,{}^1, \dots, u\,{}^n} \cdots \binom{i_n}{u_n{}^1, \dots, u_n{}^n} \\ &\qquad \times (b_{1,1}x_1)^{u_1{}^1} \cdots (b_{n,1}x_n)^{u_1{}^n} \cdots (b_{1,n}x_1)^{u_n{}^1} \cdots (b_{n,n}x_n)^{u_n{}^n} \\ &= (b_{1,1}x_1 + \cdots + b_{n,1}x_n)^{i_1} \cdots (b_{1,n}x_1 + \cdots + b_{n,n}x_n)^{i_n}. \end{aligned}$$

We have proved

PROPOSITION 9. *Let $(f_1, \ldots, f_n)$ be the delta set given by*

$$f_i = a_{i,1}t_1 + \cdots + a_{i,n}t_n$$

for $i = 1, \ldots, n$. Let $(b_{i,j}) = (a_{i,j})^{-1}$. Then in the special case $c_k = k!$ for all nonnegative integers k the associated sequence to $(f_1, \ldots, f_n)$ is

$$p_{i_1,\ldots,i_n} = (b_{1,1}x_1 + \cdots + b_{n,1}x_n)^{i_1} \cdots (b_{1,n}x_1 + \cdots + b_{n,n}x_n)^{i_n}.$$

If $f \in \Lambda$ is of the form

$$f = a_1t_1 + \cdots + a_nt_n + g,$$

where $g = 0$ or g is a power series of degree two, we call

$$\mathscr{L}(f) = a_1t_1 + \cdots + a_nt_n$$

the *linear part* of f.

THEOREM 13 (Transfer Formula). *Let $(f_1, \ldots, f_n)$ be a delta set, with $f_i = \mathscr{L}(f_i) + g_i$. Then the associated sequence for $(f_1, \ldots, f_n)$ is*

$$p_{i_1,\ldots,i_n} = \sum_{k_1,\ldots,k_n \geqslant 0}^{k_j \leqslant i_j} \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1+k_1} \cdots c_{i_n+k_n}} \det(a_{i,j})^{-1}\, \partial(f_1, \ldots, f_n)$$

$$\times \binom{-1-i_1}{k_2} \cdots \binom{-1-i_n}{k_n} g_1^{k_1} \cdots g_n^{k_n} r_{i_1+k_1,\ldots,i_n+k_n},$$

where $r_{j_1,\ldots,j_n}$ is the associated sequence for the delta set $(\mathscr{L}(f_1), \ldots, \mathscr{L}(f_n))$.

Proof. Suppose

$$f_i = a_{i,1}t_1 + \cdots + a_{i,n}t_n + g_i = \mathscr{L}(f_i) + g_i.$$

Let μ be the continuous automorphism of Λ defined by

$$\mu\mathscr{L}(f_i) = t_i$$

for $i = 1, \ldots, n$. Then the set $(\mu f_1, \ldots, \mu f_n)$ is a diagonal delta set. Therefore, it has an associated sequence in P given by Theorem 6,

$$q_{i_1,\ldots,i_n} = \frac{c_{i_1} \cdots c_{i_n}}{c_{-1}^n} \partial(\mu f_1, \ldots, \mu f_n)(\mu f_n)^{-1-i_1} \cdots (\mu f_n)^{-1-i_n} x_1^{-1} \cdots x_n^{-1},$$

where the action is of the type described in Section 6.

Now

$$\langle (\mu f_1)^{j_1} \cdots (\mu f_n)^{j_n} \mid q_{i_1,\ldots,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$$

for all integers $j_i \geqslant 0$, but any terms in $q_{i_1,\ldots,i_n}$ with negative exponents contribute nothing to this action whenever $j_i \geqslant 0$. Therefore, if we write $\tilde{q}_{i_1,\ldots,i_1}$ to denote $q_{i_1,\ldots,i_n}$ with all terms containing negative exponents removed, we obtain

$$\langle (\mu f_1)^{j_1} \cdots (\mu f_n)^{j_n} \mid \tilde{q}_{i_1,\ldots,i_n} \rangle = c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n}$$

for all integers $j_i \geqslant 0$. Moreover, we have

$$\begin{aligned} \langle f_1^{j_1} \cdots f_n^{j_n} \mid \mu^* \tilde{q}_{i_1,\ldots,i_n} \rangle &= \langle \mu f_1^{j_1} \cdots f_n^{j_n} \mid \tilde{q}_{i_1,\ldots,i_n} \rangle \\ &= \langle (\mu f_1)^{j_1} \cdots (\mu f_n)^{j_n} \mid \tilde{q}_{i_1,\ldots,i_n} \rangle \\ &= c_{i_1} \cdots c_{i_n} \delta_{i_1,j_1} \cdots \delta_{i_n,j_n} \end{aligned}$$

and so $\mu^* \tilde{q}_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1,\ldots,f_n)$.

Now $\mu f_j = t_j + \mu g_j$, where $\mu g_j = 0$ or μg_j is a power series of degree at least two. Therefore, thinking of μf_j as being in P, we have

$$(\mu f_j)^{-1-i_j} = \sum_{k_j \geqslant 0} \binom{-1-i_j}{k_j} t_j^{-1-i_j-k_j} (\mu g_j)^{k_j}$$

and so

$$\begin{aligned} q_{i_1,\ldots,i_n} &= \frac{c_{i_1},\ldots,c_{i_n}}{c_{-1}^n} \partial(\mu f_1,\ldots,\mu f_n) \sum_{k_1,\ldots,k_n \geqslant 0} \binom{-1-i_1}{k_1} \cdots \binom{-1-i_n}{k_n} \\ &\quad \times t_1^{-1-i_1-k_1} \cdots t_n^{-1-i_n-k_n} (\mu g_1)^{k_1} \cdots (\mu g_n)^{k_n} x_1^{-1} \cdots x_n^{-1} \\ &= \sum_{k_1,\ldots,k_n \geqslant 0} \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1+k_1} \cdots c_{i_n+k_n}} \partial(\mu f_1,\ldots,\mu f_n) \binom{-1-i_1}{k_1} \cdots \binom{-1-i_n}{k_n} \\ &\quad \times (\mu g_1)^{k_1} \cdots (\mu g_n)^{k_n} x_1^{i_1+k_1} \cdots x_n^{i_n+k_n}, \end{aligned}$$

where the action is that of Section 6. It is easy to see that

$$\begin{aligned} \tilde{q}_{i_1,\ldots,i_n} &= \sum_{k_1,\ldots,k_n \geqslant 0}^{k_j \leqslant i_j} \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1+k_1} \cdots c_{i_n+k_n}} \partial(\mu f_1,\ldots,\mu f_n) \binom{-1-i_1}{k_1} \cdots \binom{-1-i_n}{k_n} \\ &\quad \times (\mu g_1)^{k_1} \cdots (\mu g_n)^{k_n} x_1^{i_1+k_1} \cdots x_n^{i_n+k_n}, \end{aligned}$$

where now the action is the one of Λ on R described in this section.

We are left with computing $\mu^*\tilde{q}_{i_1,\ldots,i_n}$. If $g \in \Lambda$, then

$$\begin{aligned}\langle t_1^{i_1} \cdots t_n^{i_n} \mid \mu^* g x_1^{j_1} \cdots x_n^{j_n}\rangle &= \langle \mu t_1^{i_1} \cdots t_n^{i_n} \mid g x_1^{j_1} \cdots x_n^{j_n}\rangle \\ &= \langle g\mu t_1^{i_1} \cdots t_n^{i_n} \mid x_1^{j_1} \cdots x_n^{j_n}\rangle \\ &= \langle \mu[(\mu^{-1}g)\, t_1^{i_1} \cdots t_n^{i_n}] \mid x^{j_1} \cdots x_n^{j_n}\rangle \\ &= \langle (\mu^{-1}g)\, t_1^{i_1} \cdots t_n^{i_n} \mid \mu^* x_1^{j_1} \cdots x_n^{j_n}\rangle \\ &= \langle t_1^{i_1} \cdots t_n^{i_n} \mid \mu^{-1} g \mu^* x_1^{j_1} \cdots x_n^{j_n}\rangle\end{aligned}$$

and so

$$\mu^* g x_1^{i_1} \cdots x_n^{i_n} = \mu^{-1} g \mu^* x_1^{j_1} \cdots x_n^{j_n}.$$

Finally, since

$$\mu^{-1}\partial(\mu f_1, \ldots, \mu f_n) = \det(a_{i,j})^{-1}\, \partial(f_1, \ldots, f_n),$$

we have

$$\begin{aligned}\mu^*\tilde{q}_{i_1,\ldots,i_n} = &\sum_{k_1,\ldots,k_n \geq 0}^{k_j \leq i_j} \frac{c_{i_1} \cdots c_{i_n}}{c_{i_1+k_1} \cdots c_{i_n+k_n}} \det(a_{i,j})^{-1}\, \partial(f_1, \ldots, f_n) \\ &\times \binom{-1-i_n}{k_n} \cdots \binom{-1-i_n}{k_n} g_1^{k_1} \cdots g_n^{k_n} \mu^* x_1^{i_1+k_1} \cdots x_n^{i_n+k_n},\end{aligned}$$

where $\mu^* x_1^{j_1} \cdots x_n^{j_n}$ is the associated sequence for $(\mathscr{L}(f_1), \ldots, \mathscr{L}(f_n))$.

12. The Recurrence Formula

In this section we derive a useful recurrence formula for the associated sequence of a delta set.

If $p_{i_1,\ldots,i_n}$ is the associated sequence for a delta set $(f_1, \ldots, f_n)$ we define the *shift operators* associated with $p_{i_1,\ldots,i_n}$ (or with $(f_1, \ldots, f_n)$) as the set of operators denoted by $(\theta_{f_1}, \ldots, \theta_{f_n})$, where each θ_{f_j} is the continuous linear operator on R with

$$\theta_{f_j} p_{i_1,\ldots,i_n} = \frac{(i_j + 1)\, c_{i_j}}{c_{i_{j+1}}} p_{i_1,\ldots,i_{j-1},i_j+1,i_{j+1},\ldots,i_n}.$$

THEOREM 14. *The set of continuous linear operators $(w_1, \ldots, w_n)$ on R is a set of shift operators if and only if the set of adjoints $(w_1^*, \ldots, w_n^*)$ defined* *on Λ has the property that each w_j^* is a continuous, everywhere defined derivation of Λ and there exists some delta set $(f_1, ..., f_n)$ for which $w_j^* f_i = \delta_{i,j}$.*

Proof. Suppose $(w_1, ..., w_n)$ is the set of shift operators associated with the delta set $(f_1, ..., f_n)$. Then

$$\begin{aligned}\langle w_j^* f_1^{k_1} \cdots f_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle &= \left\langle f_1^{k_1} \cdots f_n^{k_n} \,\middle|\, \frac{(i_{j+1})\, c_{i_j}}{c_{i_{j+1}}}\, p_{i_1,\ldots,i_j+1,\ldots,i_n}\right\rangle \\ &= (i_j + 1)\, c_{i_1} \cdots c_{i_n} \delta_{i_1,k_1} \cdots \delta_{i_j+1,k_j} \cdots \delta_{i_n,k_n} \\ &= \langle k_j f_1^{k_1} \cdots f_j^{k_j-1} \cdots f_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle\end{aligned}$$

and so $w_j^* f_1^{k_1} \cdots f_n^{k_n} = k_j f_1^{k_1} \cdots f_j^{k_j-1} \cdots f_n^{k_n}$. Since w_j is continuous, so is w_j^* and thus $w_j^* = \partial/\partial f_j$ is a continuous, everywhere defined derivation on Λ. Also, it is clear that $w_j^* f_i = \delta_{i,j}$.

For the converse, suppose $(w_1^*, ..., w_n^*)$ is a set of continuous, everywhere defined derivations on Λ, and $w_j^* f_i = \delta_{i,j}$ for the delta set $(f_1, ..., f_n)$. Then if $p_{i_1,\ldots,i_n}$ is the associated sequence for $(f_1, ..., f_n)$ we have

$$\begin{aligned}\langle f_1^{k_1} \cdots f_n^{k_n} \mid w_j p_{i_1,\ldots,i_n}\rangle &= \langle k_{j_j} f_1^{k_1} \cdots f_j^{k_j-1} \cdots f_n^{k_n} \mid p_{i_1,\ldots,i_n}\rangle \\ &= \left\langle f_1^{k_1} \cdots f_n^{k_n} \,\middle|\, \frac{(i_j + 1)\, c_{i_j}}{c_{i_j+1}}\, p_{i_1,\ldots,i_j+1,\ldots,i_n}\right\rangle\end{aligned}$$

and so by the spanning argument

$$w_j p_{i_1,\ldots,i_n} = \frac{(i_j + 1)\, c_{i_j}}{c_{i_j+1}}\, p_{i_1,\ldots,i_j+1,\ldots,i_n}$$

and since w_j is continuous, we conclude that $(w_j, ..., w_n)$ is the set of shift operators for $(f_1, ..., f_n)$.

The chain rule for these derivations is easily established.

PROPOSITION 10. *If $(\theta_{f_1}, ..., \theta_{f_n})$ and $(\theta_{g_1}, ..., \theta_{g_n})$ are sets of shift operators, then*

$$\theta_{f_j}^* = \sum_{i=1}^{n} (\theta_{f_j}^* g_i)\, \theta_{g_i}^* .$$

Proof. This follows from the fact that $\theta_{f_j}^*$ is a continuous derivation, that any element of Λ can be written as a convergent sum in terms of the form $g_1^{k_1} \cdots g_n^{k_n}$, and that

$$\theta_{f_j}^* g_k = \sum_{i=1}^{n} (\theta_{f_j}^* g_i)\, \theta_{g_i}^* g_k .$$

We can now express one set of shift operators in terms of another.

THEOREM 15. *If* $(\theta_{f_1},\ldots,\theta_{f_n})$ *and* $(\theta_{g_1},\ldots,\theta_{g_n})$ *are sets of shift operators, then*

$$\theta_{f_j} = \sum_{i=1}^{n} \theta_{g_i}(\theta_{f_j}^* g_i).$$

Proof. If $p \in R$ and $h \in \Lambda$ we have

$$\begin{aligned}
\langle h \mid \theta_{f_h} p\rangle &= \langle \theta_{f_h}^* h \mid p\rangle \\
&= \left\langle \sum_{i=1}^{n} (\theta_{g_i}^* h)(\theta_{f_h}^* g_i) \,\middle|\, p \right\rangle \\
&= \sum_{i=1}^{n} \langle \theta_{g_i}^* h \mid (\theta_{f_h}^* g_i)\, p\rangle \\
&= \sum_{i=1}^{n} \langle h \mid \theta_{g_i}(\theta_{f_h}^* g_i)\, p\rangle \\
&= \left\langle h \,\middle|\, \sum_{i=1}^{n} \theta_{g_i}(\theta_{f_h}^* g_i)\, p \right\rangle
\end{aligned}$$

and the result follows from the spanning argument.

COROLLARY 13 (Recurrence Formula). *If* $(\theta_{f_1},\ldots,\theta_{f_n})$ *and* $(\theta_{g_1},\ldots,\theta_{g_n})$ *are sets of shift operators and if* $(f_1,\ldots,f_n)$ *has associated sequence* $p_{i_1,\ldots,i_n}$, *then*

$$\frac{(i_j+1)\, c_{i_j}}{c_{i_j+1}}\, p_{i_1,\ldots,i_j+1,\ldots,i_n} = \sum_{i=1}^{n} \theta_{g_i}(\theta_{f_j}^* g_i)\, p_{i_1,\ldots,i_n}.$$

The most useful version of the Recurrence Formula is for $c_k = k!$ for all $k \geqslant 0$ and $g_i = t_i$ for $i = 1,\ldots,n$. Then θ_{t_i} is multiplication by x_i and we have

COROLLARY 14 (Recurrence Formula). *In the case* $c_k = k!$, *for all* $k \geqslant 0$, *if* $(\theta_{f_1},\ldots,\theta_{f_n})$ *is a set of shift operators and if* $(f_1,\ldots,f_n)$ *has associated sequence* $p_{i_1,\ldots,i_n}$, *then*

$$p_{i_1,\ldots,i_j+1,\ldots,i_n} = \sum_{i=1}^{n} x_i \left(\frac{\partial t_i}{\partial f_j}\right) p_{i_1,\ldots,i_n},$$

where $\partial t_i/\partial f_j = \theta_{f_j}^* t_i$.

13. Examples and Applications

We will compute the associated and conjugate sequences for some classical examples. We will restrict our attention only to the algebras Λ and R, preferring to leave other examples to a forthcoming paper.

Most of the classical examples arise from the special case where $c_k = k!$ for all $k \geqslant 0$. However, it should be noted that this is not the only important case. In particular, the case $c_k = 1$ for all $k \geqslant 0$ leads to some very interesting results, but we must postpone a discussion of these.

For the most part our examples consist of delta sets $(f_1, ..., f_n)$ in which

$$f_j = \mathscr{L}_j h(\mathscr{L}_j),$$

where $\mathscr{L}_j = \mathscr{L}(f_j)$ is the linear part of f_j and where $h = h(T)$ is a power series in the variable T which has nonzero constant term. In this situation we may greatly simplify the Transfer Formula.

First, let us recall that the sequence $r_{j_1,...,j_n}$ was defined as the associated sequence for the delta set $(\mathscr{L}_1, ..., \mathscr{L}_n)$. Moreover, we saw that (since $c_k = k!$)

$$r_{i_1,...,i_n} = (b_{1,1}x_1 + \cdots + b_{in,}x_n)^{i_1} \cdots (b_{n,1}x_1 + \cdots + b_{n,n}x_n)^{i_n}.$$

If we write

$$r_j = b_{j,1}x_1 + \cdots + b_{j,n}x_n ,$$

then

$$r_{i_1,...,i_n} = r_1^{i_1} \cdots r_n^{i_n}$$

and

$$\begin{aligned} \mathscr{L}_j r_1^{i_1} \cdots r_n^{i_n} &= \mathscr{L}_j r_{i_1,...,i_n} \\ &= i_j r_{i_1,...,i_j-1,...,i_n} \\ &= r_1^{i_1} \cdots (i_j r_j^{i_j-1}) \cdots r_n^{i_n} \\ &= r_1^{i_1} \cdots (\mathscr{L}_j r_j^{i_j}) \cdots r_n^{i_n}. \end{aligned}$$

Let us consider the Transfer Formula in this setting. First we have

$$\frac{\partial f_j}{\partial t_i} = \frac{\partial \mathscr{L}_j}{\partial t_i} h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j) \frac{\partial \mathscr{L}_j}{\mathscr{L} t_i} = a_{j,i}(h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j))$$

and so

$$\partial(f_1, \ldots, f_n) = (\det a_{j,i}) \prod_{j=1}^{n} (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j)).$$

Also, since

$$f_j = \mathscr{L}_j + g_j = \mathscr{L}_j h(\mathscr{L}_j),$$

we see that

$$g_j^{k_j} = (Th - T)^{k_j}(\mathscr{L}_j).$$

Finally,

$$\frac{c_{i_j}}{c_{i_j+k_j}} \binom{-1-i_j}{k_j} = \frac{(-1)^{k_j}}{k_j!}$$

and

$$\sum_{k_j=0}^{i_j} \frac{(-1)^{k_j}}{k_j!} (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j))(Th - T)^{k_j}(\mathscr{L}_j)\, r_j^{i_j+k_j}$$

$$= (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j)) \sum_{k_j=0}^{i_j} \frac{(-1)^{k_j}}{k_j!} \left(\frac{Th - T}{T}\right)^{k_j} (\mathscr{L}_j)\, \mathscr{L}_j^{k_j} r_j^{i_j+k_j}$$

$$= (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j)) \sum_{k_j=0}^{i} \frac{(-1)^{k_j}}{k_j!} (h-1)^{k_j}(\mathscr{L}_j)(i_j + k_j)_{k_j} r_j^{i_j}$$

$$= (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j)) \sum_{h_j=0}^{\infty} \binom{-1-i_j}{k_j} (h-1)^{k_j}(\mathscr{L}_j)\, r_j^{i_j}$$

$$= (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j))\, h^{-1-i_j}(\mathscr{L}_j)\, r_j^{i_j},$$

where h^{-1-i_j} is a power series in T with nonzero constant term. We may write this suggestively as

$$\frac{\partial f_j}{\partial \mathscr{L}_j} h^{-1-i_j}(\mathscr{L}_j)\, r_j^{i_j}.$$

However, there is yet another useful form. It is easy to verify that if f is any power series in T, then

$$f'(\mathscr{L}_j)\, r_j^{i} = [f(\mathscr{L}_j)\, r_j - r_j f(\mathscr{L}_j)]\, r_j^{i}.$$

Therefore, we have

$$\begin{aligned}[h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j)]\, h^{-1-i_j}(\mathscr{L}_j)\, r_j^{i_j} &= h^{-i_j}(\mathscr{L}_j)\, r^{i_j} - i_j h'(\mathscr{L}_j)\, h^{-1-i_j}(\mathscr{L}_j)\, r^{i_j-1} \\ &= h^{-i_j}(\mathscr{L}_j)\, r_j^{i_j} - (h^{-i_j})'\,(\mathscr{L}_j)\, r_j^{i_j-1} \\ &= h^{-i_j}(\mathscr{L}_j)\, r_j^{i_j} - [h^{-i_j}(\mathscr{L}_j)\, r_j - r_j h^{-i_j}(\mathscr{L}_j)]\, r_j^{i_j-1} \\ &= r_j h^{-i_j}(\mathscr{L}_j)\, r_j^{-i_j-1}.\end{aligned}$$

We summarize our results in

THEOREM 16. *Let $(f_1, ..., f_n)$ be a delta set with*

$$f_j = \mathscr{L}_j h(\mathscr{L}_j)$$

for some power series $h = h(T)$ with nonzero constant term and where $\mathscr{L}_j = \mathscr{L}(f_j)$ is the linear part of f_j. Then in the case $c_k = k!$ for all $k \geqslant 0$, the associated sequence $p_{i_1,...,i_n}$ for $(f_1, ..., f_n)$ is given by

$$(1)\quad p_{i_1,...,i_n} = \prod_{j=1}^{n} (h(\mathscr{L}_j) + \mathscr{L}_j h'(\mathscr{L}_j))\, h^{-1-i_j}(\mathscr{L}_j)\, r_j^{i_j},$$

$$(2)\quad p_{i_1,...,i_n} = \prod_{j=1}^{n} r_j h^{-i_j}(\mathscr{L}_j)\, r_j^{i_j-1},$$

where

$$r_j^{\,i} = (b_{i,j}x_1 + \cdots + b_{n,j}x_n)^i$$

with $(b_{i,j})^{-1} = \partial(\mathscr{L}_1, ..., \mathscr{L}_n)$ and

$$\mathscr{L}_j r_j^{\,i} = i r_j^{i-1}.$$

We remark that a similar result holds if $(f_1, ..., f_n)$ is of the form

$$f_j = \mathscr{L}_j h_j(\mathscr{L}_j),$$

where h_j is a power series in T with nonzero constant term for each $j = 1, ..., n$.

We are now ready to begin our examples.

(1) The *forward difference delta set* is defined by

$$\begin{aligned} f_j &= e^{a_{j,1}t_1 + \cdots + a_{j,n}t_n} - 1 \\ &= e^{\mathscr{L}_j} - 1. \end{aligned}$$

To compute the associated sequence we use the Recurrence Formula. We have

$$\mathscr{L}_j = \log(1 + f_j)$$

and

$$\frac{\partial t_i}{\partial f_j} = b_{i,j} e^{-\mathscr{L}_j},$$

where $(b_{i,j}) = (a_{i,j})^{-1}$. The Recurrence Formula then gives

$$\begin{aligned} p_{i_1,\ldots,i_j+1,\ldots,i_n} &= \sum_{i=1}^{n} x_i b_{i,j} e^{-\mathscr{L}_j} p_{i_1,\ldots,i_n} \\ &= (b_{i,j}x_1 + \cdots + b_{n,j}x_n)\, e^{-\mathscr{L}_j} p_{i_1,\ldots,i_n}. \end{aligned}$$

Noticing that

$$\begin{aligned} e^{-\mathscr{L}_j} r_k^{\;i} &= \sum_{l\geqslant 0} \frac{(-1)^l}{l!} \mathscr{L}_j^l r_k^{\;i} \\ &= \sum_{l\geqslant 0} \frac{(-1)^l}{l!} \delta_{j,k}(i)_l\, r_k^{i-l} \\ &= (r_k - 1)^i\, \delta_{j,k} \end{aligned}$$

it is easy to see that

$$\begin{aligned} p_{i_1,\ldots,i_n} &= \prod_{j=1}^{n} r_j(r_j - 1) \cdots (r_j - i_j + 1) \\ &= \prod_{j=1}^{n} (b_{i,j}x_i + \cdots + b_{n,j}x_n)_{i_j}. \end{aligned}$$

We call these the *multivariate forward-difference polynomials.*

The conjugate sequence to the forward-difference delta set is easily computed from the definition and the fact that

$$\langle e^{c_1t_1+\cdots+c_nt_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle = c_1^{i_1} \cdots c_n^{i_n}.$$

We obtain

$$\begin{aligned} &\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle \\ &\quad = \sum_{k_1=0}^{j_1} \cdots \sum_{k_n=0}^{j_n} \binom{j_1}{k_1} \cdots \binom{j_n}{k_n} (-1)^{j_1+\cdots+j_n-k_1-\cdots-k_n} \\ &\qquad \times (a_{1,1}k_1 + \cdots + a_{n,1}k_n)^{i_1} \cdots (a_{1,n}k_1 + \cdots + a_{n,n}k_n)^{i_n}, \end{aligned}$$

which for $n = 1$ and $a_{i,j} = \delta_{i,j}$ is $j_1!$ times a Stirling number of the second kind. If we write this expression as $j_1! \cdots j_n!\, S(i_1, ..., i_n ; j_1, ..., j_n)$ we obtain

$$q_{i_1,\ldots,i_n} = \sum_{u=0}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u} S(i_1, ..., i_n ; j_1, ..., j_n)\, x_1^{j_1} \cdots x_n^{j_n}.$$

These are the *multivariate exponential polynomials* $\varphi_{i_1,\ldots,i_n}(x_1, ..., x_n)$.

(2) The *multivariate Abel polynomials* are the associated polynomials for the *Abel delta set*

$$\begin{aligned} f_j &= (a_{i,j}t_1 + \cdots + a_{n,j}t_n)\, e^{a_{1,j}t_1+\cdots+a_{n,j}t_n} \\ &= \mathscr{L}_j e^{\mathscr{L}_j}. \end{aligned}$$

In this case $h(T) = e^T$ and part (2) of Theorem 16 gives

$$A_{i_1,\ldots,i_n}(x_1, ..., x_n) = \prod_{j=1}^{n} r_j e^{-i_j \mathscr{L}_j} r_j^{i_j-1}.$$

Since

$$\begin{aligned} e^{-i_j \mathscr{L}_j} r_j^{i_j-1} &= \sum_{k \geqslant 0} \frac{(-i_j)^k}{k!} (\mathscr{L}_j)^k\, r_j^{i_j-1} \\ &= \sum_{k=0}^{i_j-1} \binom{i_j - 1}{k} (-i_j)^k\, r_j^{i_j-1-k} \\ &= (r_j - i_j)^{i_j-1}, \end{aligned}$$

we have

$$\begin{aligned} &A_{i_1,\ldots,i_n}(x_1, ..., x_n) \\ &\quad = \prod_{j=1}^{n} (b_{i,j}x_1 + \cdots + b_{n,j}x_n)(b_{i,j}x_1 + \cdots + b_{n,j}x_n - i_j)^{i_j-1}. \end{aligned}$$

The conjugate Abel polynomials are computed from the definition. They are

$$q_{i_1,\ldots,i_n} = \sum_{u=0}^{i_1+\cdots+i_n} \sum_{j_1+\cdots+j_n=u} \frac{\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle}{j_1! \cdots j_n!}\, x_1^{j_1} \cdots x_n^{j_n},$$

where

$$\langle f_1^{j_1} \cdots f_n^{j_n} \mid x_1^{i_1} \cdots x_n^{i_n} \rangle = \sum_{\substack{k_1^1 + \cdots + k^{n1} = j_1 \\ \vdots \\ k_1^n + \cdots + k_n^n = j_n}} \left(\prod_{i,j} a_{i,j}^{k_j^i} \right) (i_1)_{k_1^1 + \cdots + k_1^n} \cdots (i_n)_{k_n^1 + \cdots + k_n^n}$$
$$\times (a_{1,1} j_1 + \cdots + a_{n,1} j_n)^{i_1 - k_1^1 - \cdots - k_1^n} \cdots$$
$$\times (a_{1,n} j_1 + \cdots + a_{n,n} j_n)^{i_n - k_n^1 - \cdots - k_n^n}.$$

(3) The *multivariate Laguerre polynomials* are the associated polynomials for the *Laguerre delta set*

$$f_j = \frac{a_{i,j} t_1 + \cdots + a_{n,j} t_n}{a_{1,j} t_1 + \cdots + a_{n,j} t_n - 1} = \frac{\mathscr{L}_j}{\mathscr{L}_j - 1}.$$

In this case $h(T) = (T - 1)^{-1}$, and part (2) of Theorem 16 gives

$$L_{i_1, \ldots, i_n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} r_1^j (\mathscr{L}_j - 1)^{i_j} r^{i_j - 1}.$$

Since $(\mathscr{L}_j - 1)^{i_j} r_j^k = e^{r_j} \mathscr{L}_j^{i_j} e^{-r_j} r_j^k$ we obtain the multivariate version of the classical Rodrigues formula:

$$L_{i_1, \ldots, i_n}(x_1, \ldots, x_n) = \prod_{j=1}^{n} (b_{i,j} x_1 + \cdots + b_{n,j} x_n)$$
$$\times e^{b_{i,j} x_1 + \cdots + b_{n,j} x_n} (a_{j,1} t_1 + \cdots + a_{j,n} t_n)^{i_j}$$
$$\times e^{-(b_{1,j} x_1 + \cdots + b_{n,j} x_n)} (b_{i,j} x_1 + \cdots + b_{n,j} x_n)^{i_j - 1}.$$

From part (1) of Theorem 16 we obtain

$$L_{i_1, \ldots, i_n}(x_1, \ldots, x_n) = (-1)^n \prod_{j=1}^{n} (\mathscr{L}_j - 1)^{i_j - 1} r_j^{i_j}$$
$$= \prod_{j=1} \sum_{k_j = 1}^{i_j} \binom{i_j - 1}{k_j - 1} \frac{i_j!}{k_j!} (-1)^{k_j} (b_{i,j} x_1 + \cdots + b_{n,j} x_n)^{k_j}.$$

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REFERENCES

1. J. CIGLER, Sequences of polynomials of binomial type and the Lagrange–Good formula, preprint.
2. A. M. GARCIA AND S. A. JONI, Higher dimensional polynomials of binomial type and formal power series inversion, *Comm. Algebra* **6** (1978), 1187–1215.
3. I. J. GOOD, Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes, *Proc. Cambridge Philos. Soc.* **56** (1960), 367–380.
4. S. A. JONI, Lagrange inversion in higher dimensions and umbral operators, *J. Linear Multilinear Algebra* **6** (1978), 111–121.
5. T. MUIR, A Treetise on the Theory of Determinants, Dover publications, 1960.
6. D. L. REINER, Multivariate sequences of binomial type, *Studies in Appl. Math.* **57** (1977), 119–133.
7. S. M. ROMAN, The algebra of formal series, *Advances in Math.* **31** (1979), 309–329; erratum, in press.
8. S. M. ROMAN, The algebra of formal series II: Sheffer requences, *J. Math. Anal. Appl.*, in press.
9. S. M. ROMAN AND G.-C. ROTA, The umbral calculus, *Advances in Math.* **27** (1978), 95–188.
10. G C. ROTA, "Finite Operator Calculus," Academic Press, New York, 1975.